Multi-Armed Bandits for Selection of Risk-Based Portfolios with Transaction Costs

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Abstract

Considered the fundamental ground for modern portfolio theory, Markowitz's *Mean Variance Optimal* (MVO) portfolio is still widely used in portfolio management. However, due to its sensitivity to the estimated parameters used as input, it is limited to the time periods or market states where it can be successfully used. To counteract its limitations, after the 2008 nancial crisis, several risk-based portfolio allocation models started to be developed and used, given their lower sensitivity to the estimated parameters used as input. However, due to the large number of portfolio allocation models available, the investor is faced with a problem where she has to select one from the many available.

In this dissertation, the EXP3 and EXP3.S algorithms, originally designed to solve *Multi-Armed Bandit* (MAB) problems, are used to solve the problem in question. The investor needs only to select a set of portfolio allocation models initially, and the MAB algorithm will act as a portfolio strategy, selecting for each allocation period the \optimal" model from the set of models initially selected by the investor. The experimental results for the markets tested showed that if the investor aims to have a portfolio with a positive equilibrium between return, risk and risk diversi cation, then EXP3 and EXP3.S reveal themselves as sound compromise solutions for relatively low proportional transaction costs.

Keywords

Risk-Based Portfolios, Multi-Armed Bandits, Proportional Transaction Costs.

Resumo

Considerada a base para a teoria de portfolios moderna, o portfolio de *Variância Media Otima* (VAO) desenvolvido por Markowitz continua a ser bastante utilizado em gestão de portfolios. No entanto, devido a sua sensibilidade relativamente aos parâmetros estimados utilizados como entrada, este encontra-se limitado sobre os per odos temporais ou estados de mercado onde pode ser utilizado com sucesso. Por forma a contrariar as suas limitacões, apos a crise nanceira de 2008, varios modelos de alocacão de portfolio baseados em risco comecaram a ser desenvolvidos e utilizados dada a menor sensibilidade que estes apresentam sobre os parâmetros estimados utilizados como entrada. No entanto, devido ao elevado numero de modelos de alocacão de portfolio dispon veis, a investidora encontra-se perante um problema sobre o qual existe a necessidade de escolher um dos varios existentes.

Como tal, nesta dissertacão, os algoritmos EXP3 e EXP3.S, originalmente concebidos para resolver problemas de *Multi-Armed Bandit* (MAB), são utilizados para resolver o problema em questão. A investidora necessita apenas de selecionar inicialmente um conjunto de modelos de alocacão de portfolio, sendo que o algoritmo MAB ira actuar como uma estrategia de portfolio, selecionando para cada per odo de alocacão o modelo \otimo" do conjunto de modelos selecionados inicialmente pela investidora. Os resultados experimentais para os mercados testados mostraram que, caso a investidora pretenda ter uma carteira com um equil brio positivo entre retorno, risco e diversi cacão de risco, então o EXP3 e EXP3.S revelam-se como boas solucões de compromisso para custos de transaccão proporcionais relativamente baixos.

Palavras Chave

Portfolios Baseados em Risco, *Multi-Armed Bandits*, Custos de Transacão Proporcionais.

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Acronyms

AB	*Adversarial Bandit*
EW	*Equal Weighted*
ERC	*Equal Risk Contribution*
EWR	*Expected Weak Regret*
EWCR	*Expected Worst-Case Regret*
EXP3	*Exponential-weight algorithm for Exploration and Exploitation*
EXP3.S	*Exponential-weight algorithm for Exploration and Exploitation for Switching best strategy*
DR	*Diversi cation Ratio*
MRC	*Marginal Risk Contribution*
MVO	*Mean Variance Optimal*
MV	*Minimum Variance*
MD	*Most Diversi ed*
MAB	*Multi-Armed Bandit*
PCR	*Principal Component selection based of RMT*
QP	*Quadratic Programming*
RMT	*Random Matrix Theory*
SQP	*Sequential Quadratic Programming*
TRC	*Total Risk Contribution*

Chapter 1

Introduction

1.1 Motivation

An investor, while dealing with the financial markets, faces a decision-making problem where, being endowed with a certain initial capital, she has to determine the optimal strategy to allocate her wealth among a set of financial assets (which constitute a portfolio), over a time horizon and under market uncertainty (i.e., without knowing the assets' future returns). This problem is commonly known as the portfolio allocation problem, and appears in the last step of the four-step process of portfolio construction, which may be described as follows:

1. Select the asset classes to be included in the portfolio (asset class selection).
2. Distribute the weights for each asset class (asset class allocation policy).
3. Select the assets from each asset class that will constitute the portfolio (assets selection).
4. Distribute individual weights to the assets within each asset class that belongs to the portfolio according to the investor's preferences (portfolio allocation).

The first two steps regarding the type and weight of each asset class are generally handled by individual investors, which mainly depend on their appetite for risk and the investment's goal. For example, risk-averse investors with the aim of capital preservation prefer to allocate a more significant portion of their investment portfolio to lower-risk assets such as those in fixed-income markets (e.g. bonds). On the other hand, risk-taking investors with a primary objective of capital appreciation may prefer to allocate a more significant chunk of their wealth to stocks. On another note, the last two steps, with the goal of adding a more significant value to the investment portfolio, are often executed by taking advantage of quantitative methods. Since this dissertation is only devoted to the last step, it is assumed that all the assets under consideration are stocks, thus resulting in a portfolio only composed of one asset class. Additionally, the selection of each asset is made manually according to the investor's criteria [46].

Numerous studies have appeared in the literature to tackle the portfolio allocation problem. It started with the seminal paper \Portfolio Selection" [39], published in 1952, by Markowitz, on which he provided the first quantitative approach to portfolio construction, which became the foundation of the so-called *Modern Portfolio Theory*. He defined the process by saying that \the investor does (or should) consider expected return a desirable thing and the return's variance an undesirable one", to which he added that the efficient portfolio is the portfolio that maximises the expected return for a given level of risk (i.e., the variance of the portfolio returns). Ultimately, Markowitz concluded that there is a set of optimal portfolios, which he called the efficient frontier (for a comprehensive review, see [16]).

Later, in 1956, Markowitz introduced the *Mean Variance Optimal* (MVO)[1] portfolio allocation model [40], which has as objective the maximisation of the expected returns and the minimisation of risk (where, once again, the risk was defined as the variance of expected returns). However, despite being fairly successful during the years that followed its publication, its solution is very sensitive to the quality of the estimation of the first two moments[2] (especially the first one) [9] [17] [26] [28]. In order to deal with this problem, one could, for example, use portfolio allocation models that do not depend simultaneously on the first two moments or use estimation methods of the first two moments less prone to estimation errors.

As for the first alternative method suggested for the MVO, after the great financial crisis of 2008, risk-based strategies have become increasingly popular, which only require as input the covariance matrix of returns, where among these the *Minimum Variance* (MV), *Equal Risk Contribution* (ERC) [37] and *Most Diversified* (MD) [10] portfolios stand out. In addition, the *Equal Weighted* (EW) portfolio could also present itself as a viable alternative as one does not have to go through the estimation of the first two moments since this portfolio allocation model does not take as input either the first or second moments.

Also, since risk-based portfolios still depend on the estimation quality of the covariance matrix of returns, it is necessary to select an estimator that produces reliable and high-quality solutions. Unfortunately, despite being well known for its simplicity, the sample estimator (a non-parametric one) does not provide a reliable estimation of the covariance matrix. Therefore, within portfolio theory, other approaches have been developed, such as the shrinkage estimators assembled by Ledoit and Wolf [32] [33] [34], or factor estimators, especially the *Principal Component selection based on RMT* (PCR) estimator [30], which uses internal factors to estimate the covariance matrix (for a comprehensive review, see [12]).

Moreover, given the significant number and diversity of portfolio allocation models available to use, the investor has to decide which one is going to be used to allocate her wealth on a set of assets. This might be challenging, as it is usually the case that the various models depend on the market where they will be used and its corresponding state, as well as the assets that were selected to constitute the portfolio. For example, a given model A may present a better performance than model B in one market; however, in a different market, the opposite might occur. Thus, to consider all the possible environments and scenarios in which a portfolio allocation model will be used, the need to utilise a methodology that adapts accordingly becomes pertinent.

Thus, given the particularity of the problem that the investor faces, this is an authentic representation of a type of sequential decision-making under uncertainty that often arises in learning problems of the statistical and machine learning areas and is represented within a broad class of problems in reinforcement learning that go by the name of *Multi-Armed Bandit* (MAB). This problem concerns a situation where

[1] This and the following terms in italic are defined throughout the study.
[2] The first and second moments are the asset's expected returns and covariance matrix.

the decision-maker (the investor) is given a set of actions (the portfolio allocation models) with unknown reward distributions (e.g., the portfolio return) and has to decide which action to select at each time step in order to maximise the cumulative reward (maximise her cumulative returns). Thus, while going through it, the decision-maker has to find a balance between exploitation (i.e., selecting the currently known best action) to maximise the obtained reward and exploration (i.e., selecting an action that has to be selected more often) to maximise the amount of information that has been gathered. As one of the simplest illustrations of a reinforcement learning problem, the MAB has been used in several studies over the past decades while having an extensive practical use case from clinical trials [4], to recommendation systems [50] and telecommunications [5].

In the earliest work in the MAB literature, this problem was considered in a casino environment, where a gambler had to choose which of the available slot machines to play. At each time step, he had to pull one of the machines' actions from which a reward would be received, with these being assumed to be sampled from a fixed and unknown probability distribution independently from the previous actions (selected actions). Different solutions have been proposed for this problem, based on a stochastic formulation [2], [29], and on an adversarial one [3].

In the field of portfolio allocation, stochastic solutions to MAB problems have already been incorporated on several occasions. For example, the Upper Confidence Bound algorithm was used to perform online portfolio selection by constructing an orthogonal portfolio [45]. Another study used Thompson Sampling in an online portfolio algorithm to mix two different portfolio allocation models [44]. More recently, the same method was also used to select the portfolio allocation model at each time step, where its rewards are sampled from a Beta distribution, which is updated after each time step [51].

In this dissertation, the objective of using strategies that solve MAB problems will be to select the \optimal" portfolio allocation model at each *allocation period*, i.e., a time step in which a portfolio allocation model is used to compute the weights of the portfolio's assets, where the reward under consideration will be the normalised portfolio return. Furthermore, given that returns do not follow any particular distribution, the use of an adversarial solution becomes appropriate as it makes no statistical assumptions on the distribution of rewards, which will contribute to surpassing the limitation of the study conducted in [51], on which a distribution for the rewards was assumed. In particular, the algorithms used will be *Exponential-weight algorithm for Exploration and Exploitation* (EXP3) and *Exponential-weight algorithm for Exploration and Exploitation for Switching best strategy* (EXP3.S) [3], where the former assumes, fundamentally, the existence of a single best action, while the latter assumes that the best action can change over the time horizon under consideration. In fact, the EXP3.S may prove to be helpful since the financial markets are non-stationary, and therefore the best action may also change over time, and as such, the use of an algorithm that adapts to changes proves to be essential.

Thus, from this study it is intended to determine the following:

- *Can the EXP3 and EXP3.S algorithms achieve a performance equal or superior to the average of the various portfolio allocation models available at its disposal?*
- *Is there any di erence in performance between EXP3 and EXP3.S? Can these be applied in di erent markets?*
- *Are the models presented limited by the proportional transaction costs of constantly reallocating the portfolio's weights? If so, is it possible to overcome this issue? How?*

1.2 Overview of the Thesis

In addition to this introductory chapter, this dissertation presents in its structure ve additional chapters. These are the following:

- Chapter 2 gives the necessary mathematical background regarding the construction of risk minimisation portfolios. More speci cally, in Section 2.1, the reader is introduced to the fundamental knowledge regarding the nancial market. Then, in Section 2.2, it is given an overview of some risk minimisation portfolio allocation models. Last, in Section 2.3, it is shared some covariance estimation models, which are used as input for the optimisation problem of the selected portfolio allocation model;
- Chapter 3 introduces the MAB problem in Section 3.1. Then, in Section 3.2, it addresses the *Adversarial Bandit* (AB) problem and the algorithms used to solve it;
- Chapter 4 describes how the topics described in the previous two chapters join together, for which the backtesting process of a *portfolio strategy* is described, Section 4.2. In addition, in Section 4.1, the data type used during the backtesting is addressed. Last, in Section 4.3, it is de ned the various metrics used to evaluate the performance of a given portfolio strategy after having been through the backtesting process;
- Chapter 5 discusses and analyses the results obtained for the various portfolio strategies on real data. In particular, in Section 5.1, these strategies are evaluated on a standard set of parameters. Furthermore, their ability to perform in a market where proportional transaction costs are considered is also tested. In Section 5.2, two di erent solutions for decreasing the impact of proportional transaction costs on the investor's returns are proposed. Finally, in Section 5.3, it is analysed if the obtained best solution in the previous section can be generalised in markets other than the one used in the previous two sections;
- Chapter 6 gives a conclusion regarding the work that has been done and suggests future research directions. The appendices that follow this chapter hold complementary results and the constituents that take part in the various portfolios that were used.

Chapter 2

A framework for Risk-Based Portfolios

2.1 The Market

Consider a set of N assets $S = \{1, 2, \ldots, N\}$. Let the current time be denoted by 0, and assume a multi-period investment in a discrete-time market, i.e., the investor's wealth is re-allocated only at finitely many time steps, which constitute the set $\mathcal{T} = \{1, \ldots, T\}$, where the time horizon is set at fixed value $T \in \mathbb{N}$. Then, while re-allocating her wealth, the investor goes through a decision process during which she may sell and/or buy a certain amount of one or more assets or, on the other hand, she may choose not to take any action over the current amount of each asset in her portfolio, with all of it being entirely decided on the basis of which portfolio allocation model is being used.

Furthermore, it is assumed that the investor's wealth is fully invested in the set of N assets for all $\mathcal{T}$, commonly called as *self-financing* portfolio, which can be articulated through the notion of weights as follows,

$$\sum_{i=1}^{N} w_{i;t} = 1; \quad \forall t \in \mathcal{T}; \tag{2.1}$$

or in vector notation,

$$\mathbf{1}_N^\top \mathbf{w}_t = 1; \quad \forall t \in \mathcal{T}; \tag{2.2}$$

where $\mathbf{1}_N$ represents a N-dimension vector with each component having a value of one, and $w_{i;t}$ the wealth proportion, or weight, invested in the i-th asset at time $t \in \mathcal{T}$. Then, the weight allocated in each one of the assets that constitute the portfolio at time t is given in vector form by,

$$\mathbf{w}_t = \begin{bmatrix} w_{1;t} & w_{2;t} & \cdots & w_{N;t} \end{bmatrix}^\top; \tag{2.3}$$

where $\top$ corresponds to the vector's transpose.

Assuming that each asset that constitutes the portfolio pays no dividends[1], the *linear return* of the i-th asset between time step $t-1$ and t is defined by,

$$r_{i;t} = \frac{p_{i;t} - p_{i;t-1}}{p_{i;t-1}} = \frac{p_{i;t}}{p_{i;t-1}} - 1; \tag{2.4}$$

[1] A dividend is the weighted return on the number of shares that the investor owns from the company's profits in a given time period.

where $p_{i,t}$ and $p_{i,t-1}$ are the price of the i-th asset at time step t and $t-1$, respectively. Then, the assets' return vector at time step $t \in \mathcal{T}$ is defined as follows,

$$\mathbf{r}_t = \begin{bmatrix} r_{1,t} & r_{2,t} & \cdots & r_{N,t} \end{bmatrix}^\top. \tag{2.5}$$

Usually, in practice, the logarithmic return is the one that is often more used. The first reason is that the linear returns are bounded below by -1 and unbounded above, which brings problems when used in statistical models. On the other hand, the logarithmic returns can take values in the whole real domain, $\mathbb{R}$, which is a desirable property for modelling. The second reason is that linear returns empirically exhibit an asymmetric distribution with high skewness. On the contrary, after transforming linear returns into logarithmic ones, the former becomes more symmetric, leading to distributions that are easier to model. Finally, the third reason is that while computing the returns over multiple time steps, the logarithmic returns have the advantage of being additive over time, whereas the compounding linear returns are obtained by running a product formulation.

Despite these reasons supporting the use of the logarithmic returns, the linear ones have the desirable property for portfolio allocation by being additive over assets. Also, since the magnitude of the value returned by both types of returns is usually small, using linear or logarithmic returns turns out to be approximately equal for most investment scenarios. Therefore, given that the study under consideration resides in the subject of portfolio allocation, the returns will be assumed to be linear [49]. So then, the linear returns will be referred to as returns from now on.

Furthermore, conditional on the set of historical returns up to time t, $\mathcal{F}_t$, the assets' *expected returns* are given by,

$$\boldsymbol{\mu}_t = \mathbb{E}\left[\mathbf{r}_{t+1} \mid \mathcal{F}_t\right], \tag{2.6}$$

and the *covariance matrix* by,

$$\boldsymbol{\Sigma}_t = \mathbb{E}\left[\left(\mathbf{r}_{t+1} - \boldsymbol{\mu}_t\right)\left(\mathbf{r}_{t+1} - \boldsymbol{\mu}_t\right)^\top \mid \mathcal{F}_t\right]. \tag{2.7}$$

Moreover, the portfolio return for a single time step $t \in \mathcal{T} \setminus \{1\}$ is given by,

$$R_t^p = \sum_{i=1}^{N} w_{i,t-1} r_{i,t}, \tag{2.8}$$

or in vector notation,

$$R_t^p = \mathbf{w}_{t-1}^\top \mathbf{r}_t, \tag{2.9}$$

with $R_1^p = 0$, as the first allocation occurs at $t = 1$. Then, the portfolio returns over a time horizon T

are then given by the following vector,

$$\mathbf{R}^p = \left(R_1^p \cdots R_T^p \right)^\top . \tag{2.10}$$

Furthermore, the *portfolio's expected return* at time step $t \in \mathcal{T}$ is given by,

$$\mu_p(\mathbf{w}_t) = \mathbb{E}\left[R_{t+1}^p \right] = \mathbf{w}_t^\top \boldsymbol{\mu}_t , \tag{2.11}$$

and the respective *variance* by,

$$\sigma_p^2(\mathbf{w}_t) = \mathbb{E}\left[\left(R_{t+1}^p - \mu_p(\mathbf{w}_t) \right) \left(R_{t+1}^p - \mu_p(\mathbf{w}_t) \right)^\top \right] = \mathbf{w}_t^\top \boldsymbol{\Sigma}_t \mathbf{w}_t , \tag{2.12}$$

where it is assumed as the portfolio's risk metric the standard deviation, or *volatility*, which at time step $t \in \mathcal{T}$ is defined as,

$$\sigma_p(\mathbf{w}_t) = \sqrt{\mathbf{w}_t^\top \boldsymbol{\Sigma}_t \mathbf{w}_t} . \tag{2.13}$$

2.2 Risk-Based Portfolios

This section introduces four risk-based portfolio allocation models from the literature: the equal weighted, minimum variance, equal risk contribution and most diversified portfolios. A broader analysis can be found in [15].

2.2.1 Equal Weighted Portfolio (EW)

As the name suggests, the *Equal Weighted* (EW) portfolio consists of the equal allocation of wealth in each of the N assets that constitute the portfolio. Then, the weight of the i-th asset at time step $t \in \mathcal{T}$ is given by,

$$w_{i,t} = \frac{1}{N} , \tag{2.14}$$

for $i = 1, \ldots, N$.

Thus, given its characteristics, the EW portfolio can be considered the most diversified portfolio regarding the distribution of the weight's values, having the most evenly spread amount of wealth invested in each asset. On the other hand, given that the EW portfolio allocation model does not take into account the individual risk generated by each asset, it may lead to a portfolio that does not have a balanced distribution of risk per asset.

In addition, the EW portfolio is often considered a benchmark for having better performance in terms of *Sharpe ratio*[2] and *turnover*[3] than portfolio allocation models that rely on the goodness of the

[2] The *Sharpe ratio* measures the reward-to-risk ratio of a portfolio strategy. It is defined formally in Section 4.3.1.

[3] The *turnover* measures the average proportion of wealth that has been traded for a given period. It is defined formally in Section 4.3.2.

estimation of the first two moments [14].

2.2.2 Minimum Variance Portfolio (MV)

The idea of the portfolio theory laid by Markowitz had its fundamental ground in finding the correct trade-off between the expected return and the portfolio risk measured by the variance [39]. These desired characteristics were used to construct the *Mean Variance Optimal* (MVO) portfolio [40], where its formulation consists of minimising the return-adjusted risk. Then, at time step $t \in \mathcal{T}$, the minimisation problem is given as follows,

$$\begin{aligned} \min_{\mathbf{w}_t} \ & \frac{1}{2}\sigma_p^2(\mathbf{w}_t) - \lambda\mu_p(\mathbf{w}_t) \\ \text{s.t.} \ & \mathbf{w}_t^\top \mathbf{1}_N = 1 \end{aligned}, \tag{2.15}$$

with λ being the risk-aversion parameter, where the lower its value, the higher the investor's priority in minimising the risk.

In the case of $\lambda = 0$, the problem given in (2.15) gives rise to the formulation used by the portfolio called *Minimum Variance* (MV), which only minimises the portfolio variance while avoiding the usage of the expected returns as input. Then, the MV portfolio at time step $t \in \mathcal{T}$ may be formulated by the following *Quadratic Programming* (QP) problem with equality constraints,

$$\begin{aligned} \min_{\mathbf{w}_t} \ & \frac{1}{2}\sigma_p^2(\mathbf{w}_t) \\ \text{s.t.} \ & \mathbf{w}_t^\top \mathbf{1}_N = 1 \end{aligned}, \tag{2.16}$$

for which by solving the *Karush-Kuhn-Tucker* (KKT) optimality conditions [6] directly yields the closed-form solution expressed as,

$$\mathbf{w}_t = \frac{\Sigma_t^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma_t^{-1}\mathbf{1}}. \tag{2.17}$$

In practice, additional constraints are often added to portfolio optimisation problems, which might be due to market regularisation or investors' preferences. The most common one is the so-called *long-only* constraint, which ensures that the weight of each asset cannot be lower than zero (assuming short-selling is not allowed, one cannot sell what one does not have). This one will then be considered in the portfolio allocation models that follow, as it has significant efficacy in improving portfolios constructed under the estimation of the first two moments [25].

Then, adding the long-only constraint to the problem described in (2.16), the new minimisation problem at time step $t \in \mathcal{T}$ becomes,

$$\begin{aligned} \min_{\mathbf{w}_t} \ & \frac{1}{2}\sigma_p^2(\mathbf{w}_t) \\ \text{s.t.} \ & \mathbf{w}_t^\top \mathbf{1}_N = 1, \\ & \mathbf{w}_t \geq \mathbf{0} \end{aligned} \tag{2.18}$$

for which no closed-form solution is available, and therefore, it must be solved using numerical methods.

Furthermore, it has been found that the MV portfolio presents a good out-of-sample[4] performance in a complete market cycle[5]. On the other hand, since the model diversifies the asset's weights according to their variance, the portfolio often shows to be highly concentrated on a small number of assets. In fact, portfolios that only use as input the covariance matrix often share this drawback.

2.2.3 Equal Risk Contribution Portfolio (ERC)

Formally introduced by Maillard [37], the *Equal Risk Contribution* (ERC) portfolio, also known as the *Risk Parity* (RP) portfolio, was created as an alternative to the two other portfolio allocation models discussed previously (EW and MV). It was designed during a new trend of portfolio strategies that surged after the 2008 financial crisis, aiming at diversifying the portfolio through an optimal distribution of risk, rather than the weight amount, among the assets. In this model, each asset that belongs to a portfolio contributes equally to its risk, where the wealth attributed to each asset is the same only if all assets have the same volatility. On the other hand, assets that have a lower risk or lower correlation (or both) with other assets are assigned a larger weight in the portfolio.

In order to formulate how one can compute the ERC portfolio, two concepts should be defined beforehand. First, the change in the portfolio's risk induced by an infinitesimal increase/decrease in the weight of an asset is given by its *Marginal Risk Contribution* (MRC). Then, at time step $t \in \mathcal{T}$, the MRC of the i-th asset is given by,

$$\partial_{w_{i,t}} \sigma_p(\mathbf{w}_t) = \frac{\partial \sigma_p(\mathbf{w}_t)}{\partial w_{i,t}} = \frac{(\Sigma_t \mathbf{w}_t)_i}{\sigma_p(\mathbf{w}_t)}, \tag{2.19}$$

where $(\Sigma_t \mathbf{w}_t)_i$ is the i-th component of the vector $\Sigma_t \mathbf{w}_t$.

The second concept refers to the *Total Risk Contribution* (TRC) of an asset, which corresponds to the proportion of the total portfolio risk attributed to the asset. Then, at time step $t \in \mathcal{T}$, the TRC of the i-th asset is given as follows,

$$\sigma_i(\mathbf{w}_t) = w_{i,t} \partial_{w_{i,t}} \sigma(\mathbf{w}_t) = \frac{w_{i,t}(\Sigma_t \mathbf{w}_t)_i}{\sigma_p(\mathbf{w}_t)}, \tag{2.20}$$

from which the following decomposition is obtained,

$$\sigma_p(\mathbf{w}_t) = \sum_{i=1}^{N} \sigma_i(\mathbf{w}_t); \quad \forall t \in \mathcal{T}, \tag{2.21}$$

where the sum of the TRC of each asset equals the portfolio's total risk, i.e., the portfolio's volatility.

[4] The data points used during the computation of the portfolio weights constitute *in-sample* data, whereas all the new points left out constitute *out-of-sample* data. For example, stating that a portfolio has a good/bad out-of-sample performance means that the model can have a good/bad performance in data points it did not use during its training.

[5] A *complete economic cycle* corresponds to a market that goes through both bull (when the market is rising in value, and the economic conditions are favourable) and bear (when the market is declining in value paralleled with an economy that is receding) phases.

Then, formally, the ERC portfolio should satisfy the following equality,

$$\sigma_i(\mathbf{w}_t) = \sigma_j(\mathbf{w}_t) \Leftrightarrow w_{i,t}(\Sigma_t \mathbf{w}_t)_i = w_{j,t}(\Sigma_t \mathbf{w}_t)_j; \quad \forall i, j \in S; \forall t \in T. \tag{2.22}$$

That is, the risk contribution of the i-th asset should be the same as the risk contribution of the j-th asset at time step $t \in T$.

In order to get a portfolio with these properties, it is required to use numerical algorithms since a closed-form solution does not exist. There are several different specific formulations on the ERC portfolio. However, before diving into these, a general one is going to be defined, and only afterwards, a connection between the general and some existing specific formulations is provided. Thus, the general ERC problem at time step $t \in T$ can be expressed as,

$$\begin{aligned} \min_{\mathbf{w}_t} \quad & RC(\mathbf{w}_t) \\ \text{s.t.} \quad & \mathbf{w}_t^\top \mathbf{1}_N = 1, \\ & \mathbf{w}_t \geq 0 \end{aligned} \tag{2.23}$$

where,

$$RC(\mathbf{w}_t) = \sum_{i=1}^{N} \left(g_i(\mathbf{w}_t) \right)^2, \quad \forall t \in T, \tag{2.24}$$

is a measure of the portfolio's risk concentration, with $g_i(\mathbf{w}_t)$ being a smooth differentiable non-convex function that measures the risk concentration of the i-th asset at time step $t \in T$ [20].

Furthermore, diving into the specific formulations, the original approach took advantage of a *Sequential Quadratic Programming* (SQP) algorithm, with the portfolio weights at time step $t \in T$ being obtained by solving the following minimisation problem,

$$\begin{aligned} \min_{\mathbf{w}_t} \quad & \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_{i,t}(\Sigma_t \mathbf{w})_{i,t} - w_{j,t}(\Sigma_t \mathbf{w})_{j,t} \right)^2 \\ \text{s.t.} \quad & \mathbf{w}_t^\top \mathbf{1}_N = 1 \\ & \mathbf{w}_t \geq 0 \end{aligned} . \tag{2.25}$$

The formulation given in (2.25) revolves around minimising the variance of the (re-scaled) risk contributions by penalising the summation of squared differences among risk contributions, with the existence of an ERC portfolio only being ensured if the equality in (2.22) is valid.

Later, Bruder and Roncalli [8] proposed a slight modification to the objective function in order to accelerate the computation time[6]. The minimisation problem at time step $t \in T$ may then be formulated

[6] By using this formulation it is possible to compute analytically the associated gradient and Hessian matrices

as follows,

$$
\begin{aligned}
\min_{\mathbf{w}_t} \quad & \sum_{i=1}^{N} \left(\frac{w_{i,t}(\boldsymbol{\Sigma}_t \mathbf{w}_t)_i}{\sigma_p^2(\mathbf{w}_t)} - \frac{1}{N} \right)^2 \\
\text{s.t.} \quad & \mathbf{w}_t^\top \mathbf{1}_N = 1 \\
& \mathbf{w}_t \geq \mathbf{0}
\end{aligned} \quad . \tag{2.26}
$$

Despite both minimisation problems (2.25) and (2.26) being solvable while using non-linear numerical optimisation methods (e.g. SQP), the type of problem formulation used is frequently not well explored by this family of optimisation methods, with its computation being often time-consuming. Additionally, the obtained solution may not be of high quality.

In order to have reliable solutions, Feng [20] proposed a new family of algorithms that turn the non-convex formulation of the ERC portfolio into a convex one by using an optimisation tool which goes by the name of successive convex optimisation methods. The proposed method uses an iterative procedure, where at the k-th iteration and time step $t \in \mathcal{T}$, it aims at solving the following problem,

$$
\begin{aligned}
\min_{\mathbf{w}_t} \quad & P(\mathbf{w}_t; \mathbf{w}_t^{(k)}) + \frac{\tau}{2} \|\mathbf{w}_t - \mathbf{w}_t^{(k)}\|_2^2 \\
\text{s.t.} \quad & \mathbf{w}_t^\top \mathbf{1}_N = 1 \\
& \mathbf{w}_t \geq \mathbf{0}
\end{aligned} \quad ; \tag{2.27}
$$

where,

- $\mathbf{w}_t^{(k)}$ is the vector of the portfolio weights for the k-th iteration, with its values being updated with the solution $\mathbf{w}_t$ obtained after solving the minimisation problem (its use case will be clearer when Algorithm 1 gets introduced);

- $P(\mathbf{w}_t; \mathbf{w}_t^{(k)})$ is a convex quadratic function that corresponds to the approximation of the term $RC(\mathbf{w}_t)$ and is obtained by linearising each $g_i(\mathbf{w}_t)$ contained inside the square operation in $RC(\mathbf{w}_t)$. It is expressed as follows,

$$
P(\mathbf{w}_t; \mathbf{w}_t^{(k)}) = \sum_{i=1}^{N} \left(g_i(\mathbf{w}_t^{(k)}) + \left(\nabla g_i(\mathbf{w}_t^{(k)}) \right)^\top (\mathbf{w}_t - \mathbf{w}_t^{(k)}) \right)^2 ; \quad \forall t \in \mathcal{T} ; \tag{2.28}
$$

- $\frac{\tau}{2} \|\mathbf{w}_t - \mathbf{w}_t^{(k)}\|_2^2$ is the regularisation term added for convergence purposes [43], where τ is a given non-negative constant, and $\|\cdot\|_2$ is the Euclidean norm.

The formulation given in (2.27) generalises the problems defined in (2.25) and (2.26) while having a convex objective function. However, the term inside the square operation in the last two mentioned problems must present a new structure to make the problem in (2.27) feasible. Thus, according to the new formulation, at time step $t \in \mathcal{T}$, the new structure for the term inside the square operation of the problem given in (2.26) is given as follows,

$$g_i(\mathbf{w}_t) = \frac{\mathbf{w}_t^\top \mathbf{M}_{i,t}\mathbf{w}_t}{\sigma_p^2(\mathbf{w}_t)}, \tag{2.29}$$

and its gradient as,

$$\nabla g_i(\mathbf{w}_t) = \frac{(\mathbf{w}_t^\top \Sigma_t \mathbf{w}_t)(\mathbf{M}_{i,t} + \mathbf{M}_{i,t}^\top)\mathbf{w}_t - (\mathbf{w}_t^\top \mathbf{M}_{i,t}\mathbf{w}_t)(2\Sigma_t)\mathbf{w}_t}{\sigma_p^4(\mathbf{w}_t)}, \tag{2.30}$$

where $\mathbf{M}_{i,t} \in \mathbb{R}^{N \times N}$ denotes the sparse matrix with its i-th row being the same as that of the covariance matrix Σ_t and 0 elsewhere.

Then, $P(\mathbf{w}_t; \mathbf{w}_t^{(k)})$ can be rearranged in order to have a more compact expression, which leads to the following,

$$P(\mathbf{w}_t; \mathbf{w}_t^{(k)}) = \|\mathbf{A}_t^{(k)}(\mathbf{w}_t - \mathbf{w}_t^{(k)}) + \mathbf{g}(\mathbf{w}_t^{(k)})\|_2^2, \quad \forall t \in T, \tag{2.31}$$

where,

$$\mathbf{A}_t^{(k)} = \left[\nabla g_1(\mathbf{w}_t^{(k)}) \quad \cdots \quad \nabla g_N(\mathbf{w}_t^{(k)})\right]^\top, \quad \forall t \in T, \tag{2.32}$$

$$\mathbf{g}(\mathbf{w}_t^{(k)}) = \left[g_1(\mathbf{w}_t^{(k)}) \quad \cdots \quad g_N(\mathbf{w}_t^{(k)})\right]^\top, \quad \forall t \in T. \tag{2.33}$$

Then, for a given time step $t \in T$, the problem in (2.27) can be rewritten as follows,

$$\begin{aligned} \min_{\mathbf{w}_t} \quad & \frac{1}{2}\mathbf{w}_t^\top \mathbf{Q}_t^{(k)}\mathbf{w}_t + \mathbf{w}_t^\top \mathbf{q}_t^{(k)} \\ \text{s.t.} \quad & \mathbf{w}_t^\top \mathbf{1}_N = 1 \\ & \mathbf{w}_t \geq 0 \end{aligned}, \tag{2.34}$$

where,

$$\mathbf{Q}_t^{(k)} = 2(\mathbf{A}_t^{(k)})^\top \mathbf{A}_t^{(k)} + \tau \mathbf{I}, \quad \forall t \in T, \tag{2.35}$$

$$\mathbf{q}_t^{(k)} = 2(\mathbf{A}_t^{(k)})^\top \mathbf{g}(\mathbf{w}_t^{(k)}) - \mathbf{Q}_t^{(k)}\mathbf{w}_t^{(k)}, \quad \forall t \in T. \tag{2.36}$$

The sequential solving approach, based on a successive convex optimisation methodology, is represented in Algorithm 1, which is executed on each time step $t \in T$. In addition, taking into consideration the theoretical framework developed in [43], it can be shown [20] that under some technical assumptions and $\tau > 0$, $\gamma^k \in (0, 1]$, $\gamma^k \to 0$, $\sum_k \gamma^k = +\infty$, and $\sum_k (\gamma^k)^2 < +\infty$ that either Algorithm 1 converges in a finite number of iterations to a stationary point $\mathbf{w}_t$, $\forall t \in T$, or every limit of $\mathbf{w}^{(k)}$ (at least one such point exists) is a stationary point $\mathbf{w}_t$, $\forall t \in T$.

Moreover, the ERC portfolio is considered to be less sensitive to small changes in the estimated covariance matrix than the MV portfolio. Additionally, if the correlation and Sharpe ratio are equal among all the assets that constitute the portfolio, the ERC and MV portfolios are the exactly the

Algorithm 1: Successive Convex optimisation for RIsk Parity portfolio (SCRIP)

Input: $w_t^{(0)}; \; > 0; \; ^k > 0; \; > 0.$

for $k = 0; 1; \dots$ **do**

1. Compute w_t by solving the minimisation problem in (2.34), which takes as input $w_t^{(k)}$.
2. $w_t^{(k+1)} = w_t^{(k)} + \; ^k(w_t \quad w_t^{(k)})$.
3. **if** $\|w_t^{(k+1)} \quad w_t^{(k)}\|_1 \quad \frac{1}{2} \quad \|w_t^{(k+1)}\|_1 + \|w_t^{(k)}\|_1$ **then**
   - Output the stationary point w_t.

   else
   - Continue.

same [37]. As previously stated, the ERC portfolio can be seen as an intermediary between the MV and EW portfolios, as it is a form of variance-minimising portfolio subject to a constraint of sufficient diversification in terms of component weights. Then, the portfolio volatility of these three portfolio allocation models can be ordered in the following way,

$$\sigma_p(w_t)^{MV} \leq \sigma_p(w_t)^{ERC} \leq \sigma_p(w_t)^{EW}; \quad \forall t \in T;$$

for which the respective proof is available in [37, Appendix B].

2.2.4 Most Diversified Portfolio (MD)

Taking the properties of diversification as a criterion in portfolio allocation, Choueifaty and Coignard [10] introduced the *Most Diversified* (MD) portfolio, which uses a measure of the degree of portfolio diversification, the *Diversification Ratio* (DR),

$$DR(w_t) = \frac{w_t^{\top} \sigma_t}{\sigma_p(w_t)}; \quad \forall t \in T; \tag{2.37}$$

where the numerator is the weighted average volatility of the individual assets in the portfolio and the denominator is the portfolio volatility. Then, in other words, the DR gives the ratio between the weighted average portfolio volatility and the portfolio volatility.

Additionally, the DR measure tries to capture the nature of diversification laid by Markowitz (on which the diversification translates into decreasing the overall portfolio risk by multiple assets [18]), in that the overall portfolio volatility should be less than the weighted average volatility of the individual assets if the portfolio was properly diversified. For a long-only portfolio, in order for it to be well-diversified, the DR measure should give a value larger than 1.

Then, for a time step $t \in T$, the MD portfolio is obtained by solving the following non-convex maximisation problem,

$$
\begin{aligned}
\max_{\mathbf{w}_t} \quad & DR(\mathbf{w}_t) \\
s.t. \quad & \mathbf{w}_t^{\top} \mathbf{1}_N = 1, \\
& \mathbf{w}_t \geq 0
\end{aligned}
\tag{2.38}
$$

which can be solved by using a SQP algorithm.

Moreover, it was shown in [11] that given a convex set of constraints, since the DR is invariant by scalar multiplication, the MD portfolio optimisation can be formulated as a QP problem. The solution for it exists, and it is unique if the covariance matrix is definite. Then, for a time step $t \in T$, the problem is given by,

$$
\begin{aligned}
\min_{\mathbf{w}_t} \quad & \frac{1}{2} \sigma_p^2(\mathbf{w}_t) \\
s.t. \quad & \mathbf{w}_t^{\top} \sigma_t = 1, \\
& \mathbf{w}_t \geq 0
\end{aligned}
\tag{2.39}
$$

where the resulting weights are subsequently re-scaled in order for these to sum up to 1 when added.

Furthermore, the MD nature can be understood by its two core properties [11], which are the following:

- Any asset not held by the MD is more correlated to the MD than any of the assets that belong to it. Furthermore, all assets belonging to the MD have the same correlation to it.

- The long-only MD is the long-only portfolio such that the correlation between any other long-only portfolio and itself is greater than or equal to the ratio of their DRs.

The first core property implies that all assets in the investment universe are effectively represented in the MD, even if their weight value is equal to zero. For example, an MD portfolio composed of 100 assets may only attribute a weight to 20 of them, leaving the remaining with a weight of zero. This is due to the fact that the 20 assets that it actually holds are less correlated to the MD than the other 80 that it does not hold. In other words, MD gives more weight to assets that show less correlation with the others. For example, assume two different assets, A and B, which have a correlation of 1 between each other and share the same correlation with all the other assets in the portfolio, except with asset C. Then, between A and B, the one that shows less correlation with C is attributed a higher weight in the portfolio. On the other hand, if a given asset has a correlation of 1 with all the other assets in the portfolio, it is given a zero weight as the other assets are already representing it.

The second core property implies that the more a long-only portfolio is diversified, the higher the correlation of the portfolio's returns with that of the MD.

At last, the MD portfolio is equal to the MV portfolio if all N assets have the same volatility. Additionally, the MD portfolio is optimal if the assets' expected returns are proportional to their volatilities.

2.3 Covariance Matrix Estimator

Considering the last three portfolio allocation models described (MV, ERC and MD), each one takes the covariance matrix of the asset's returns as input. However, given that the respective covariance matrix is not known at the time that the optimisation takes place, it has to be estimated. Therefore, this leaves the portfolio allocation model highly dependent on the covariance matrix's estimation model being used. If this one is more error-prone, it may lead to a non-optimal distribution of weights on the portfolio's assets. Consequently, it leads to an increase in the volatility of returns and a higher turnover [12], resulting in a higher expenditure of the investor's wealth in transaction costs, lowering the returns of her investment. As such, up next, two different proposals for the covariance matrix estimator are presented.

Sample Estimator

Given its simplicity, the sample estimator is often used. Then, while using T_o historical returns, with $T_o \in \{1, \dots, T\}$, the sample covariance matrix[7] at time step $t \in T$ is given as follows,

$$\hat{\Sigma}_t = \frac{1}{T_o - 1} \sum_{t'=t-T_o} (\mathbf{r}_{t'} - \bar{\mathbf{r}}_{t'})(\mathbf{r}_{t'} - \bar{\mathbf{r}}_{t'})^\top; \quad \forall t \in T; \tag{2.40}$$

where the sample mean returns are given by,

$$\bar{\mathbf{r}}_t = \frac{1}{T_o} \sum_{t'=t-T_o} \mathbf{r}_{t'}; \quad \forall t \in T. \tag{2.41}$$

Despite its popularity, it usually presents a high estimation error, especially when the ratio between the number of assets N that constitute the portfolio and the number of observations T_o available to perform the estimation is significant. Thus, the ratio N/T_o plays a determinant role in the precision of the sample estimator. In addition, when this ratio presents a high value, the estimation error of the covariance matrix tends to be larger than the one of the expected value [19] [27]. However, when $N < T_o$, the sample covariance matrix can be used, in which case the matrix would be non-singular. Nonetheless, in practice, when the values of N and T_o are close, it leads to an ill-conditioned matrix, which is close to singular [12].

Given the constraints presented by the sample estimator, several alternatives have been studied in the literature. A study conducted by Coqueret [12] was made to test which state-of-the-art covariance matrix estimator was optimal in a portfolio allocation setting. In this one, he compared the various estimators according to their volatility and turnover results. These were tested in different markets and for different sample sizes of historical returns T_o. According to the obtained results, it was concluded that one estimator stood out from the rest, as its performance was either the best one or among the top four in the various tests, and therefore it will be described next.

[7] The Bessel's correction is used in order to correct the bias in the estimation of the portfolio's return variance.

Principal Component selection based of RMT Estimator

Proposed initially by Laloux [30], with extensions being found in Akemann [1], derived solely from historical returns, with no assumptions being made on the connection between the returns and external data, the *Principal Component selection based of RMT* (PCR) estimator takes advantage of a result obtained in *Random Matrix Theory* (RMT) [38] to make a robust estimation of the covariance matrix that is suitable for financial data.

Then, for a selected number of historical returns $T_o \in \mathcal{T}$, the PCR's estimation procedure starts with the eigendecomposition of the sample correlation matrix,

$$\hat{\Omega}_t = \mathrm{P}_t \Lambda_t \mathrm{P}_t^{\top}; \quad \forall t \in \mathcal{T}; \tag{2.42}$$

where, $\Lambda_t = \mathrm{diag}(\lambda_{1;t}, \ldots, \lambda_{N;t})$, is the diagonal matrix of eigenvalues, with $\lambda_{1;t} \geq \cdots \geq \lambda_{N;t}$, and P_t is the matrix of the normalised eigenvectors.

Then, one has to decide how many factors/eigenvalues should be kept/selected in order to have a balance between explanatory power and statistical significance. From a result obtained in RMT [38], it states that when observations are independent and have the same stationary distribution with unit variance[8] and the ratio N/T_o converges to a constant q different from 1, then, asymptotically, all non-zeros eigenvalues of $\hat{\Omega}_t$ will lie in the interval,

$$I = [(1 - \sqrt{q})^2; (1 + \sqrt{q})^2]; \tag{2.43}$$

with this result only holding when both N and T_o tend to infinity. In practice, this result is applied in finite samples, with the estimated value of q being equal to N/T_o. Then, the upper bound of the interval I is estimated as follows,

$$\lambda_{N;T_o} = \left(1 + \sqrt{\frac{N}{T_o}}\right)^2. \tag{2.44}$$

Furthermore, when dealing with financial data, it is often the case that the largest eigenvalue from the sample correlation matrix lies considerably above the estimated upper bound given in (2.44). This can be explained by the fact that the investment universe is strongly driven by the "market factor", which accounts for a substantial proportion of the total variance of the universe (e.g., [7] shows that in the US stock market, it accounts for more than 75%). Additionally, depending on the number of assets that belong to the portfolio, a few other eigenvalues may lie outside the theoretical bounds of the interval given in (2.43) [12].

The gap between the theoretical and empirical distribution of the eigenvalues is then viewed as relevant information. At the same time, the eigenvalues that lie within the theoretical bounds are viewed

[8] This result is applied in the context of the eigendecomposition of the correlation matrix since it is assumed that all assets have unit variance.

as simply noise, and therefore their information is not considered to be useful [30]. Thus, taking this into consideration, the new eigenvalues are given according to the following expression[9],

$$\tilde{\lambda}_{i,t} = \begin{cases} \lambda_{i,t} & \text{if } \lambda_{i,t} > \lambda_{N,T_o} \\ \dfrac{\sum_{j=1}^{N} \lambda_{i,t} \mathbb{1}_{\{\lambda_{i,t} \leq \lambda_{N,T_o}\}}}{N_{\text{low}}} & \text{otherwise} \end{cases}, \quad \forall t \in \mathcal{T}, \tag{2.45}$$

where the eigenvalues which have values above the upper bound λ_{N,T_o} are maintained, while the ones that have their value below or equal to this upper bound (these being at most $N_{\text{low}} \leq N$) are individually assigned the average value of all the eigenvalues that share the same condition [1] [30] (these new eigenvalues will be the eigenvalues of sample correlation matrix in (2.47)). This selecting procedure of the eigenvalues results in a decrease of the amplitude of the off-diagonal elements of the sample covariance matrix, while maintaining its diagonal values.

Then, the sample correlation matrix is then rewritten as follows,

$$\hat{\Omega}_t = P_t \tilde{\Lambda}_t P_t', \quad \forall t \in \mathcal{T}, \tag{2.46}$$

where, $\tilde{\Lambda}_t = \text{diag}\left(\tilde{\lambda}_{1,t}, \cdots, \tilde{\lambda}_{N,t}\right)$, is the new diagonal matrix of the eigenvalues obtained via (2.45). After the computation of $\hat{\Omega}_t$, its diagonal values are all set to one, which, together with the following computation given in (2.47), ensures that the resulting covariance matrix is non-singular, that the eigenvalues are less dispersed than those of the sample covariance matrix, and that the sample variances are preserved [12].

At last, the PCR estimation of the covariance matrix is obtained by,

$$\hat{\Sigma}_t = \hat{\sigma}_t \hat{\Omega}_t \hat{\sigma}_t, \quad \forall t \in \mathcal{T}, \tag{2.47}$$

where, $\hat{\sigma}_t = \text{diag}\left(\hat{\sigma}_{1,1,t}, \cdots, \hat{\sigma}_{N,N,t}\right)$, is the diagonal matrix of the assets' sample volatility.

Thus, having just described the PCR estimation model, in the following chapters, it is assumed that the covariance matrix estimation is done while using this very model.

[9] Here, $\mathbb{1}_{\{A\}}$ denotes the *indicator function*, which is defined by,

$$\mathbb{1}_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Chapter 3

A framework for the Multi-Armed Bandit Problem

Consider a sequential decision problem defined by a finite and fixed set of available actions between a randomised player and an adversary. At each time step, the player chooses an action to perform, after which a numerical reward, assigned by the adversary, is obtained. The player's goal is to maximise the expected total reward over a sequence of decisions. Four versions of this problem are typically considered:

- *Full Information* problem: at the end of each time step, the player observes the reward value assigned to each action;

- *Label Efficient* problem: after selecting an action at a given time step, the player decides whether to ask for the rewards of the different actions at this time step, knowing that this can only be done for a limited number of times [24];

- *Multi-Armed Bandit* (MAB) problem: the player only observes the reward obtained from the selected action, but not the reward values of the remaining actions [41];

- *Label Efficient Bandit* problem: the only observed rewards are the ones that were obtained and asked by the player, with again a limited number of possible queries [23].

Then, from the four problems described above, the MAB one will be studied in more detail since the type of problem it represents and its assumptions align with the one dealt within this dissertation. Its name was given by analogy to a slot machine, or "one-armed bandit"; however, in this case, the gambler faces multiple instead of only facing one arm. Thus, in this circumstance, the MAB problem can be seen as a gambler playing several slot machines at the same time while trying to select the one that will return the highest payoff [47].

Moreover, since it is assumed that the player can only observe the reward obtained from the selected action at each time step, the MAB problem suffers from limited information. Therefore, one must address the fundamental trade-off between exploration and exploitation in a sequence of decision periods. On the one hand, the player needs to *explore*: to try out other actions to acquire new information; on the other hand, the player has to *exploit*: to make optimal decisions based on the current information. Therefore, the player must balance the number of times she exploits actions that have yielded positive results in the past and the number of times she explores actions that may yield positive results in the future.

3.1 The Problem

For an arbitrary time horizon $T > 0$, with $I = \{1, \ldots, T\}$ constituting the set of the T time steps, the MAB problem is specified by the number K of possible actions, where each action is included in the set,

$$A = \{1, \ldots, K\}. \tag{3.1}$$

In addition, the reward vector at each time step $t \in I$ is given as follows,

$$\mathbf{x}_t = \begin{bmatrix} x_{1,t} & \cdots & x_{K,t} \end{bmatrix}^\top, \tag{3.2}$$

with the T vectors that were computed for the whole duration of the time horizon being included in the set,

$$X = \{\mathbf{x}_1, \ldots, \mathbf{x}_T\}. \tag{3.3}$$

The player's selected actions up to time step $t \in I$ are given by the following vector,

$$\mathbf{a}_t = \begin{bmatrix} a_1 & \cdots & a_t \end{bmatrix}^\top, \tag{3.4}$$

where $a_t \in A, \forall t \in I$. Additionally, the player's cumulative reward at time step $t \in I$ is given by,

$$G_{\mathbf{a}_t,t} = \sum_{t'=1} x_{a_{t'},t'}, \quad \forall t \in I, \tag{3.5}$$

which holds for any assignment of rewards.

In order to formalise a MAB problem, there are several regimes that can be considered. However, in this dissertation, only the following two will be addressed:

- *Stochastic regime*: the rewards are generated independently at random, ruled by fixed but unknown probability distributions with means μ_i for each action $i \in A$ (first considered by Robbins [41]);

- *Adversarial regime*: the rewards are selected arbitrarily by an adversary, and therefore no statistical assumption is made regarding their distribution (first considered by Auer [3]).

Given the data used in Chapter 5, the adversarial regime is selected to be the one to be used moving forward, as its rewards by nature do not make any statistical assumptions regarding the distribution of the data being used. Thus, a more detailed theoretical description of how the adversarial regime works will be given in the next section. The algorithms used to solve MAB problems in this particular regime are also included.

3.2 Adversarial Bandits

The *Adversarial Bandit* (AB) problem, as previously stated, is a variant of the MAB problem in which no statistical assumptions are made about the generation of rewards. The problem may be described as an iterated three-step process:

1. the adversary picks the rewards distribution;
2. the player selects an action without awareness of the adversary's selections;
3. the rewards are assigned.

Furthermore, without any constraints, the AB can be seen as a competition between the player and an omniscient adversary with unlimited computational power and memory capacity to stay ahead of any strategy the player selects. Thus, within this regime, three di erent types of adversaries are built from the constraints applied to the adversary's selection process. Fundamentally, allowing the distribution to vary in each time step, the following distinction is made:

- *Fully-Oblivious*: the rewards are independent, both through time and across actions, and independent of the player's decisions. Thus, the adversary is characterised by T distributions for each action (the adversaries in the stochastic regime are an example of fully-oblivious adversaries);
- *Oblivious*: the only constraint on the adversary is that the rewards are independent of the player's previous selected actions;
- *Non-Oblivious*: the adversary may choose the rewards at time step t as a function of the player's previous selected actions $\boldsymbol{a}_{t\ 1}$.

Then, considering the various types of adversaries, since the rewards used in practice have no dependency on the actions previously selected by the player, it is assumed that the adversary is Oblivious. Additionally, the adversary is not considered to be Full-Oblivious, since the algorithms used to solve the AB problem are built in a way that the reward vectors are not independent, and also, the rewards obtained in practice are not independent over time or between actions.

Lastly, several algorithms have been developed to solve the AB problem. In this dissertation, two of those will be addressed, with each one having been selected for a di erent reason:

1. *EXP3 algorithm*: assumes that, for a time horizon T, there exists an \optimal" action. I.e., it is assumed that there exists one action superior to all the others for the whole time horizon. It was designed so that the ratio between the considered superior action and all the other increases with the number of time steps taken;
2. *EXP3.S algorithm*: assumes that the \optimal" action can change more than once over T. In other words, there is no assumption about the existence of a superior action over the whole

time horizon. In fact, it takes into consideration that there may be switches between the action that is considered superior (the one that has been given the highest reward for a given time period).

3.2.1 EXP3 Algorithm

The simplest algorithm within the adversarial regime is the so-called *Exponential-weight algorithm for Exploration and Exploitation* (EXP3) [3], described in Algorithm 2. It was developed based on the *Hedge* algorithm [21], which was used for solving a different worst-case sequential decision problem.

Algorithm 2: Exponential-weight algorithm for Exploration and Exploitation (EXP3)

Parameters: $\gamma \in (0,1] \subset \mathbb{R}$.
Initialisation: $w_{i,1} = 1$, for $i = 1, \ldots, K$.
For $t = 1, \ldots, T$ **do**

1. **For** $i = 1, \ldots, K$ **set**
$$p_{i,t} = (1-\gamma)\frac{w_{i,t}}{\sum_{j=1}^{|\mathcal{A}|} w_{j,t}} + \frac{\gamma}{K}.$$
2. Draw action a_t randomly according to the probability distribution vector $\mathbf{p}_t$.
3. Receive reward $x_{a_t,t} \in [0,1]$.
4. **For** $i = 1, \ldots, K$ **set**
$$\hat{x}_{i,t} = \begin{cases} \frac{x_{i,t}}{p_{i,t}} & \text{if } i = a_t \\ 0 & \text{otherwise} \end{cases},$$
$$w_{i,t+1} = w_{i,t}\exp\left(\frac{\gamma \hat{x}_{i,t}}{K}\right).$$

The EXP3 algorithm is composed of an iterative process that moves with time t, for $t \in \mathcal{I}$. The algorithm is divided into four different steps. However, before executing it, an initial value must be given to an input parameter in order for it to run as expected. This parameter, $\gamma \in (0,1]$, measures the fraction of time the algorithm spends on selecting a purely random decision and is called the *learning rate*.

The vector with the associated weights for each action at time step $t \in \mathcal{I}$ is given by,

$$\mathbf{w}_t = \left[w_{1,t} \cdots w_{K,t}\right]^\top, \tag{3.6}$$

with $\mathbf{w}_1 = \mathbf{1}_K$.

At each time step $t \in \mathcal{I}$, the EXP3 starts by drawing an action $a_t \in \mathcal{A}$ according to the probability distribution vector,

$$\mathbf{p}_t = \left[p_{1,t} \cdots p_{K,t}\right]^\top, \tag{3.7}$$

with the probability of selecting action i at time step t being given by,

$$p_{i,t} = (1-\gamma)\frac{w_{i,t}}{W_t} + \frac{\gamma}{K}, \tag{3.8}$$

where,

$$W_t = \sum_{j=1}^{|\mathcal{A}|} w_{j,t}. \tag{3.9}$$

Intuitively, adding the uniform distribution in (3.8) ensures that the algorithm tries out all K actions and gets good estimates of the rewards for each. If this one is not added, the algorithm might miss a good action because the initial rewards it observes for this action are low and good rewards, that might occur later, are not observed because the action is not selected.

Furthermore, after having selected the action a_t, the reward $x_{a_t,t}$ is received, which then the EXP3 uses to compute the estimated reward $\hat{x}_{a_t,t}$, with this one being given by the ratio between the received reward $x_{a_t,t}$ and the probability that the action was selected, $p_{a_t,t}$. This is done to reward those actions that are unlikely to be chosen. Thus, this choice of estimated rewards guarantees that their expectations are equal to the actual rewards for each action, i.e.,

$$\mathrm{E}[\hat{x}_{i,t}|\mathbf{a}_{t-1}] = x_{i,t}, \quad \forall i \in \mathcal{A}, \forall t \in \mathcal{I}, \tag{3.10}$$

where the expectation is taken with respect to the random choice of a_t at time step t given the actions $\mathbf{a}_{t-1}$ selected in the previous $t-1$ time steps.

At last, the weights for the next time step $t+1$ are computed by exponential re-weighting of the cumulative rewards. The updated weight for each action $i \in \mathcal{A}$ is then given by,

$$w_{i,t+1} = w_{i,t} \exp\left(\frac{\gamma \hat{x}_{i,t-1}}{K}\right), \quad \forall t \in \mathcal{I}. \tag{3.11}$$

In order to measure the performance of the player's selected actions via EXP3, the *Expected Weak Regret* (EWR) is used as two different upper bounds on it were proven for the EXP3 algorithm. The EWR gives the difference between the cumulative reward of the best single action (i.e., the action that, by only selecting it through the time period under consideration, achieved the highest cumulative reward),

$$G_t^{(max)} = \max_{i \in \mathcal{A}} \sum_{t'=1}^{t} x_{i,t'}, \quad \forall t \in \mathcal{I}, \tag{3.12}$$

and the expected cumulative reward of the player's selected actions $\mathbf{a}_t$, thus being defined by the following expression,

$$\mathrm{EWR}_t(\mathbf{a}_t) = G_t^{(max)} - \mathrm{E}[G_{a_t,t}], \quad \forall t \in \mathcal{I}, \tag{3.13}$$

where the expectation on $G_{a_t,t}$ is taken due to the fact that at each time step the selected action is computed while using the probability distribution $\mathbf{p}_t$, defined in (3.7).

Thus, for any $K > 0$, any $T > 0$, and any $\gamma \in (0, 1]$, the following upper bound,

$$\mathrm{EWR}_T(a_T) \le (e-1)\gamma G_T^{(max)} + \frac{K \ln(K)}{\gamma}, \tag{3.14}$$

holds for any assignment of rewards [3]. On the other hand, if EXP3 selects the γ value deterministically via,

$$\gamma = \min\left\{1, \sqrt{\frac{K\ln(K)}{(e-1)g}}\right\}, \tag{3.15}$$

then, for any $T > 0$ and assuming that $g \ge G_T^{(max)}$, it follows that,

$$\mathrm{EWR}_T(a_T) \le 2\sqrt{e-1}\sqrt{gK\ln(K)}, \tag{3.16}$$

holds for any assignment of rewards [3].

To apply the result given in (3.16), an upper bound g of $G_T^{(max)}$ must exist to tune the γ value. Given that the rewards are assumed to be bounded in the interval $[0, 1]$, no action can have a reward greater than 1 at any time step, and therefore, for a known time horizon $T > 0$, the upper bound $g = T$ can be used.

3.2.2 EXP3.S Algorithm

In many contexts, including the one being studied in this dissertation, it is not reasonable to assume that the environment does not change along the learning process, and thus the use of a algorithm that takes this into account becomes relevant. Thus, the *Exponential-weight algorithm for Exploration and Exploitation for Switching best strategy* (EXP3.S) is a variant of the EXP3 algorithm, where the non-stationarity of the best action is considered, i.e., the best action in one time step may not be the best in another.

The EXP3.S algorithm, described in Algorithm 3, goes through the exact same steps as EXP3. However, the expression used to update the weights of each action is different. When computing the weights for the next time step $t + 1$, a proportion mean gain is added to the exponential re-weighting of the cumulative rewards. This addition allows the algorithm to forget about the past (allows the data to "decay" out of the algorithm), thus facilitating its behaviour to be more open to changes in the optimal action. Thus, the updated weight for each action $i \in A$ is given by,

$$w_{i,t+1} = w_{i,t} \exp\left(\frac{\gamma \hat{x}_{i,t-1}}{K}\right) + \frac{e\alpha}{K}\sum_{j=1}^{K} w_{j,t}. \tag{3.17}$$

Algorithm 3: Exponential-weight algorithm for Exploration and Exploitation for Switching best strategy (EXP3.S)

Parameters: $\gamma \in (0, 1] \subset \mathbb{R}$; $\alpha \in \mathbb{R}^+$.
Initialisation: $w_{i,1} = 1$, for $i = 1, \ldots, K.$
For $t = 1, \ldots, T$ **do**

1. **For** $i = 1, \ldots, K$ **set**

$$p_{i,t} = (1 - \gamma)\frac{w_{i,t}}{\sum_{j=1}^{|A|} w_{j,t}} + \frac{\gamma}{K}.$$

2. Draw action a_t randomly according to the probability distribution vector $\mathbf{p}_t$.
3. Receive reward $x_{a_t,t} \in [0, 1].$
4. **For** $i = 1, \ldots, K$ **set**

$$\hat{x}_{i,t} = \begin{cases} \frac{x_{i,t}}{p_{i,t}} & \text{if } i = a_t \\ 0 & \text{otherwise} \end{cases},$$

$$w_{i,t+1} = w_{i,t}\exp\left(\gamma\frac{\hat{x}_{i,t}}{K}\right) + \frac{e\alpha}{K}\sum_{j=1}^{K} w_{j,t}.$$

In order to measure the performance of the player's selected actions via EXP3.S, the *Expected Worst-Case Regret* (EWCR) is used as three different upper bounds on it were proved for the EXP3.S algorithm. Before defining these, consider any sequence of actions selected up to time step $t \in I$ given by,

$$\mathbf{s}_t = \begin{bmatrix} s_1 & \cdots & s_t \end{bmatrix}^{\top}, \tag{3.18}$$

where $s_t \in A, \; \forall t \in I$, and the following quantity,

$$\mathrm{H}_t(\mathbf{s}_t) = 1 + \sum_{t'=1}^{t-1} \mathbb{1}_{\{s_{t'} \neq s_{t'+1}\}}, \tag{3.19}$$

which gives the *hardness* of a sequence $\mathbf{s}_t$. Then, the EWRC is defined as,

$$\mathrm{EWCR}_t(\mathbf{s}_t, \mathbf{a}_t) = G_{\mathbf{s}_t,t} - \mathbb{E}[G_{\mathbf{a}_t,t}], \quad \forall t \in I, \tag{3.20}$$

where,

$$G_{\mathbf{s}_t,t} = \sum_{t'=1}^{t} x_{s_{t'},t'}, \quad \forall t \in I, \tag{3.21}$$

is the cumulative reward at time step $t \in I$, which was obtained by having selected any sequence of actions $\mathbf{s}_t$. Additionally, it should be noted that the bound on the EWCR grows with the hardness of a given sequence $\mathbf{s}_t$.

Thus, for any $K > 0$, any $T > 0$, any $\gamma \in (0, 1]$, and any $\alpha > 0$, the following upper bound,

$$\mathrm{EWCR}_T(\mathbf{s}_T, \mathbf{a}_T) \leq \frac{K(\mathrm{H}_T(\mathbf{s}_T)\ln(K/\alpha) + e\alpha T)}{\gamma} + (e - 1)\gamma T, \tag{3.22}$$

holds for any assignment of rewards, and for any sequence of actions $\mathbf{s}_T$ [3]. On the other hand, if the input parameters α and γ used by EXP3.S are determined deterministically via,

$$\alpha = \frac{1}{T}; \qquad \gamma = \min\left\{1; \sqrt{\frac{K\ln(KT)}{T}}\right\}; \tag{3.23}$$

then,

$$\text{EWCR}_T(s_T; a_T) \leq \text{H}_T(s_T)\sqrt{KT\ln(KT)} + 2e\sqrt{\frac{KT}{\ln(KT)}}; \tag{3.24}$$

holds for any sequence of actions s_T [3].

Moreover, consider S the upper bound of the hardness, i.e., $\text{H}_T(s_T) \leq S$, in case it exists. If the hardness is bounded by S, then the input parameters α and γ used by EXP3.S are then given by,

$$\alpha = \frac{1}{T}; \qquad \gamma = \min\left\{1; \sqrt{\frac{K(S\ln(KT) + e)}{(e-1)T}}\right\}; \tag{3.25}$$

for which the upper bound,

$$\text{EWCR}_T(s_T; a_T) \leq 2\sqrt{e-1}\sqrt{KT(S\ln(KT) + e)}; \tag{3.26}$$

holds for any sequence of actions s_T such that $\text{H}_T(s_T) \leq S$ [3].

Chapter 4

Formalisation of the Trading Framework

So far, the reader has only been introduced to the theoretical background needed to address the various portfolio allocation models and MAB problems. Therefore, this chapter elaborates on how these two topics connect and work together while describing the backtesting process for a portfolio strategy[1], Section 4.2. Additionally, the data type used during the backtesting and the performance metrics used to analyse the obtained results are also addressed in Sections 4.1 and 4.3, respectively.

4.1 Data Selection

In order to proceed with the testing of a given portfolio strategy on a set of historical returns, known as *backtesting*, it is first necessary to define the data type on which this takes place. Therefore, a collection of data regarding the assets that constitute a given portfolio must be selected. Depending on the investor's objective, the type of the selected assets may vary in various ways. For example, an investor may choose to diversify her portfolio between assets that present risk and others that do not, or on the other hand, she can opt to select only one of the two types. Then, if the investor chooses to have a mix of the two, given that the portfolio allocation models here studied are entirely focused on the minimisation of risk, the inclusion of risk-free assets would lead to a portfolio highly concentrated on this type of assets. At the same time, the assets with inherent risk would have no relevance whatsoever.

Thus, moving forward, it is assumed that a portfolio consists only of assets with inherent risk, which in this study are stocks, with these remaining unchanged during the totality of the time horizon T while sharing among themselves the market in which they participate[2]. Therefore, considering this, the investor initially selects the market she wants to invest in, which is subsequently followed by selecting the assets that constitute the portfolio.

Additionally, to assess the portfolio's excess return (i.e., the portfolio's return after subtracting the risk-free asset's return during the same investment period), a risk-free asset operating in the same market as the assets that constitute the investor's portfolio is selected. The risk-free asset may, for example, take the form of a:

- *Government Bond*: a debt obligation issued by a national government to support its spending. When one is bought, the government commits to the investor to pay her a periodic interest rate on

[1] The term *portfolio strategy* not only takes into account the portfolio allocation models previously described (EW, MVO, MV, ERC and MD) but also the EXP3 and EXP3.S algorithms.

[2] This restriction was made in order to facilitate the analysis that is taken in Chapter 5.

the amount bought, called coupons. At the end of the commitment period, the amount invested initially is returned to the investor. The commitment can vary depending on the type of bond the investor selects. For example, a *Treasury Bill* (bond issued by the *United States Department of the Treasury* to finance government spending) has a commitment term of less than one year;

- *London Inter-Bank Offered Rate* (LIBOR): an average interest rate calculated from the major London banks' estimates of what they would charge if they borrowed money from other banks.

Finally, one should note that the risk-free asset return is only subtracted from the portfolio return and not directly from the assets' return before the portfolio allocation has been conducted. In addition, Figure 4.1 represents the data selection procedure before the backtesting process takes place.

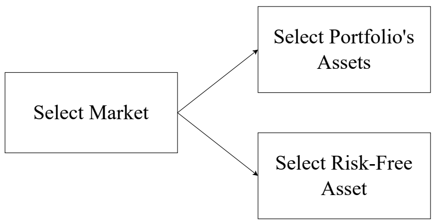


Figure 4.1: Procedure for selecting the data used in the backtesting process.

4.2 Backtesting

After being introduced to the data type used during the backtesting process, this section describes the set of parameters and the phases that compose this process. Before touching on these two, it should be noted that two different formulations are used to categorise the various portfolio strategies. The separation is made by considering the set of parameters being used and the whole process of allocating the portfolio's weights. The first, called CORE, corresponds to using a single portfolio allocation model for T, while the second, called CORE+, corresponds to using a MAB algorithm, in particular EXP3 or EXP3.S, for selecting a portfolio allocation model at each time step.

Additionally, the CORE+ formulation differs from the CORE in that the portfolio allocation model does not remain fixed over T, which means that the investor is not limited to the performance of a single portfolio allocation model. At last, from now on, it is considered that in both formulations, only MV, ERC and MD portfolio allocation models can be used[3].

4.2.1 Parameters

This section describes the parameters that are common to both CORE and CORE+, as well as those that are exclusively used in the CORE+ formulation.

[3] As addressed in Chapter 5, the EW portfolio will only be used as a benchmark, and the MVO was only previously described in order to introduce MV.

4.2.1.A Common to both formulations CORE and CORE+

Before describing the backtesting process in more detail, it is necessary to define a set of parameters used throughout this process. As such, the set of parameters common to both CORE and CORE+ formulations are the following:

- F: identifies the formulation of the portfolio strategy to be used during backtesting process, with the assignable values being CORE and CORE+. If CORE, the investor must select one of the portfolio allocation models MV, ERC or MD. On the other hand, if CORE+, the investor has to select the MAB algorithm to be utilised, EXP3 or EXP3.S;

- L: corresponds to the size of the rolling window used during the estimation of the covariance matrix. Its purpose will become more evident in Section 4.2.2.A, where the rolling window method is described;

- H: defines the time interval between two allocation periods, with $H \in \{1, \dots, T - L\}$. For instance, if this parameter has a value of 1, at each time step t, a new allocation of the portfolio weights takes place. On the other hand, if its value is set to 20, then a new allocation of the portfolio weights only takes place every 20 time steps. Additionally, if $H = T - L$, only one allocation is performed for the whole backtesting process. This is due to the fact that for $t = T - L$, no weight allocation is done since the returns that would be collected are not observable as this is the last time step;

- TC: defines the proportional transaction cost in buying or selling a given asset, with $TC \in [0, 1]$. The amount subtracted from the investor's wealth increases linearly with the amount bought or sold of a given asset. A more detailed description is given in Section 4.2.2.B;

- TO: defines the maximum absolute value of the difference in the weight of an asset between the current allocation period and at the end of the previous time step (after the assets return have been considered). Its intent becomes clarified in Section 4.2.2.B.

4.2.1.B Only for formulation CORE+

The CORE+ formulation requires extra parameters besides those described previously, which are directly associated with both EXP3 and EXP3.S algorithms. Thus, the additional parameters that are particular to this formulation are presented as follows:

- K: corresponds to the number of actions that the MAB algorithms have. Since, in this study, it has been considered that these only have a set of three actions available (each corresponding to one of the portfolio allocation models MV, ERC or MD), this parameter is set to a fixed value of $K = 3$;

- : de nes the learning rate of algorithms EXP3 and EXP3.S, where its value is assigned in order to ensure that the bound of EWR and EWCR is guaranteed, respectively. For the EXP3 algorithm, its value is given according to (3.15) and EXP3.S according to (3.23);
- : exclusive to the EXP3.S algorithm, its value is assigned such that the EWCR bound, given in (3.23), is ensured.

4.2.2 Phases

This section describes the four main phases that make up the backtesting process. These will be shared in an order that eases the comprehension of the various concepts, despite not being the actual order that these phases take during the backtesting process, Figure 4.2.

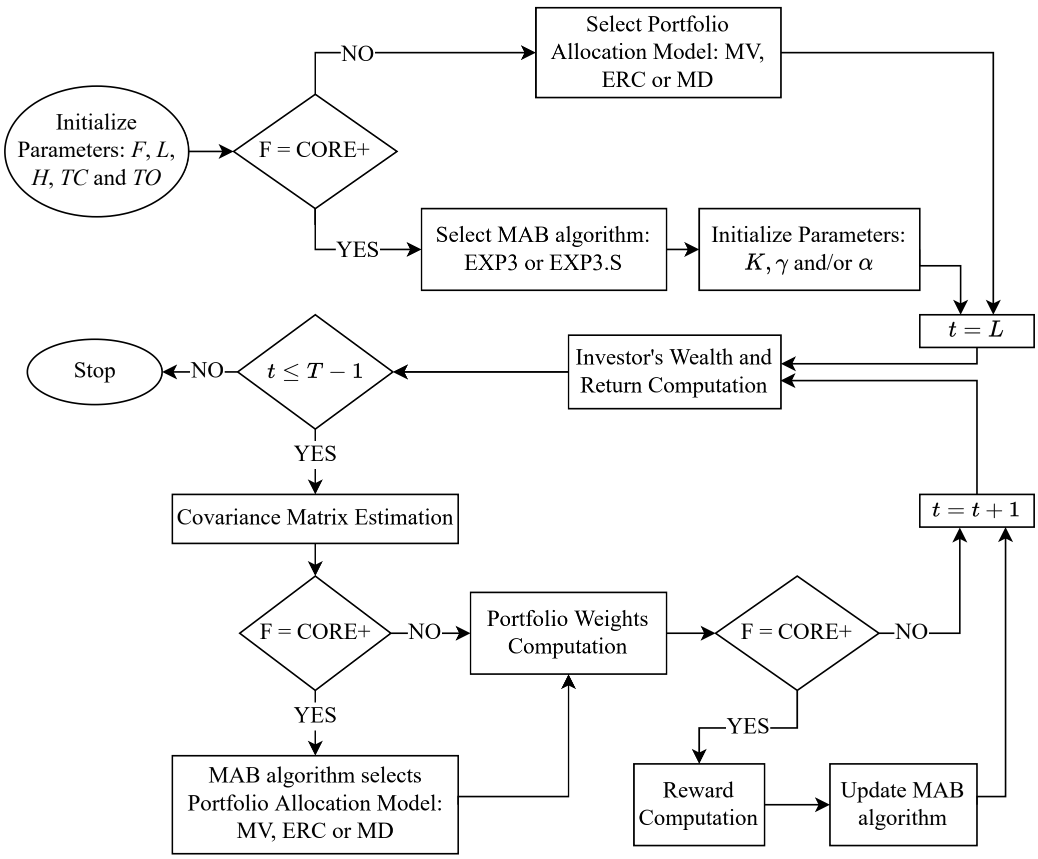


Figure 4.2: Backtesting process over a sequence of T observations.

4.2.2.A Covariance Matrix Estimation

This phase consists of estimating the covariance matrix using the PCR method de ned in Section 2.3, for which the returns of each asset are used. It starts with selecting the window over which the estimation is performed. As stated previously, the length of the estimation window is denoted by the parameter

$L < T - 1$, with T being the total number of historical returns available for each asset.

Then, by means of the *rolling window* method [14] [48], the window is updated for the next step by dropping the earliest observation of the asset's returns and adding the latest one, as in Figure 4.3. The described procedure is repeated until time step $T - 1$[4].

Despite only being needed at each allocation period, the covariance matrix estimation is done at each time step as its values are used during the computation of the effective number of correlated bets (a performance metric described in Section 4.3.2).

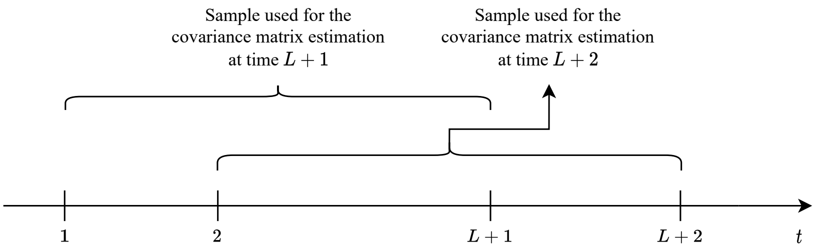


Figure 4.3: Example of the rolling window method between two consecutive time steps. Note that $t \neq 0$ since, at this time step, the return of each asset is still unknown.

4.2.2.B Portfolio Weights Computation

As the name suggests, the portfolio weights computation is where the portfolio weights are computed using one of the two different procedures available, which is selected according to the time step at hand.

On the one hand, if $(t - L) \bmod H \neq 0$, with $t \in \{L, \ldots, T - 1\}$, then it means that the current time step does not correspond to an allocation period. However, although no portfolio allocation model is put in place, the portfolio weights for a given time step t may differ from those at $t - 1$, resulting from price fluctuations of the various assets that constitute the portfolio. The weights at time step t would only remain unchanged relative to those at $t - 1$ if the price of each asset in the portfolio did not change between these two time steps. Therefore, the portfolio weight's vector at time step t is given by the entry-wise multiplication (denoted by $\odot$) between the weight's vector at $t - 1$ and the assets' return vector at t, with the respective result being normalised by the product between the transposed weight's vector at $t - 1$ and the assets' return vector at t (such that the sum of the weights is one). Then, under the described scenario, the weights are computed using the following expression [14],

$$\mathbf{w}_t = \frac{\mathbf{w}_{t-1} \odot (\mathbf{1}_N + \mathbf{r}_t)}{\mathbf{w}_{t-1}^{\top} (\mathbf{1}_N + \mathbf{r}_t)}. \tag{4.1}$$

On the other hand, if the time step corresponds to an allocation period, then portfolio weights are computed using a portfolio allocation model. If one had initially chosen the CORE formulation, then

[4] The estimation is not done at time step T since no portfolio weights are attributed at this time step, where the latter is not done because the returns that would be received from it would only be materialised at $T + 1$ which goes beyond the total number of historical observations available.

the portfolio allocation model used is the one selected. However, for CORE+, the portfolio allocation model used to compute the portfolio weights in an allocation period will depend on the MAB algorithm's action, which has the MV, ERC and MD portfolio allocation models at its disposal.

Moreover, since the MAB algorithm is only used in each allocation period, the total number of observations that EXP3 and EXP3.S consider is the number of allocation periods T_{AP} that takes place during a time horizon T, with $T_{AP} < T$. Then, the total number of allocation periods is given by,

$$T_{AP} = \left\lceil \frac{T - L}{H} \right\rceil; \tag{4.2}$$

which is used for the value attribution of both parameters and , where $\lceil \cdot \rceil$ is the ceiling function.

Regarding the TO parameter, if the value attributed is different than zero and $t \in \{L + H, \dots, T - 1\}$ (i.e., if the computation of the portfolio weights does not take place on the first allocation period), a new constraint is added to the portfolio allocation model, which is given by,

$$w_{i,t-1^+} - \frac{TO}{N} \leq w_{i,t} \leq w_{i,t-1^+} + \frac{TO}{N}; \quad \forall i \in S; \tag{4.3}$$

where, $w_{i,t-1^+} = \frac{w_{i,t-1}(1+r_{i,t})}{\mathbf{w}_{t-1}^{\top}(\mathbf{1}_N + \mathbf{r}_t)}$ is the weight of the i-th asset at time step $t-1$ before the rebalancing at time step t [14]. For example, in the case of the EW portfolio strategy, $w_{i,t} = w_{i,t-1} = 1/N$, but $w_{i,t-1^+}$ may be different due to changes in asset prices between $t-1$ and t. The notation $t-1^+$ used in $w_{i,t-1^+}, \forall i \in S$, refers to the portfolio weights at the time instance immediately before computing the weights at time step t through a portfolio allocation model, where its value reflects the weights at $t-1$ on top of the returns yielded by each asset between $t-1$ and t.

One should note that the selection of the TO value is made by taking into account the whole portfolio turnover. Therefore, when used in (4.3), its value needs to be divided by the total number of assets, N, available in the portfolio because the restriction is individual to each asset. In addition, it should be noted that the turnover described here differs from the one presented in Section 4.3.2, as if the latter was used as a constraint, it would have turned the optimisation problem into a non-convex one, whereas the former keeps the problem convex.

Furthermore, after the portfolio computation has been carried out, the proportional transaction cost for buying or selling a certain amount of a given asset is charged immediately. Then, at time step $t \in \{L + 1, \dots, T - 1\}$, the proportional transaction cost is defined as follows,

$$\text{PTC}_t = 1 - TC \sum_{i=1}^{N} \lVert w_{i,t} - w_{i,t-1^+} \rVert_1; \tag{4.4}$$

where, $\lVert \cdot \rVert_1$ is the absolute value norm. Additionally, when $t = L$, since this is the first allocation period, the value of the proportional transaction cost is then given by,

$$\text{PTC}_L = 1 - TC. \tag{4.5}$$

4.2.2.C Investor's Wealth and Return Computation

In this phase, two tasks are performed, which are directly dependent. The first one consists of the computation of the investor's wealth, W_t, at the beginning of each time step $t \in \{L, \dots, T\}$. For the first allocation period, it is considered that the investor has in her possession a positive amount of wealth, $W_L > 0$, which she can use for investing in a portfolio of assets. The second one consists in computing the investor's returns at time step $t \in \{L+1, \dots, T\}$ from the portfolio weights computed at $t-1$. Note that at $t = L$, the investor's returns cannot be obtained because the first portfolio allocation is performed in this exact time step, with its respective returns being obtained only in the following time step.

Then, computing the investor's wealth and return for $t \in \{L+1, \dots, T\}$ starts with the computation of the portfolio's excess return given its weights distribution at the previous time step. This step is necessary as the portfolio's return obtained via (2.9) does not consider the return the investor would have received if she had allocated her wealth on a risk-free asset at time step $t-1$. Thus, the portfolio's excess return at time step $t \in \{L+1, \dots, T\}$ is given by,

$$R_t^{p_e} = R_t^p - r_t^{rf}; \tag{4.6}$$

with r_t^{rf} being the return of risk-free asset at time step $t \in \{L+1, \dots, T\}$.

Thus, given the portfolio's excess return obtained through (4.6), at time step $t \in \{L+1, \dots, T\}$, the wealth of the investor is given as follows,

$$W_t = W_{t-1}(1 + R_t^{p_e}) - \text{PTC}_{t-1}; \tag{4.7}$$

and as for the investor's return, this one is given by,

$$R_t = \frac{W_t}{W_{t-1}} - 1. \tag{4.8}$$

4.2.2.D Reward Computation and MAB update

Unlike the previously described phases, this one only occurs if the chosen formulation is CORE+ and the time step corresponds to an allocation period ($t \in \{L, \dots, T-1\}$, with $(t-L) \bmod H = 0$). This phase starts with the reward computation, and then, from the obtained result, it updates the action's weight selected by the MAB algorithm, according to (3.11) or (3.17), if EXP3 or EXP3.S, respectively. Additionally, its value is within the interval $[0, 1]$, so the EWR, defined in (3.16), is guaranteed.

Thus, taking into account the EW portfolio allocation model defined in Section 2.2.1, the reward for updating the respective MAB algorithm at time step $t \in \{L+H, \dots, T-H-1\}$, with $(t-L) \bmod H = 0$ and $T_{AP} > 2$, is obtained as follows:

1. Compute all the cumulative portfolio's returns generated between the various allocation pe-

riods that have occurred up to time step t, as if the investor had used the EW portfolio allocation model instead, with the set of these returns being given by,

$$\mathcal{R}_t^{(EW)} = \left\{ R_2^{(EW)}, \ldots, R_{t_{AP}}^{(EW)} \right\}, \tag{4.9}$$

where, $t_{AP} = \frac{t-L}{H} + 1$ and,

$$R_h^{(EW)} = \prod_{i=1}^{H} \left(1 + \mathbf{r}_{t-H(t_{AP}-h+1)+i}^{\top} \mathbf{w}_{t-H(t_{AP}-h+1)+i-1}^{(EW)} \right) - 1, \tag{4.10}$$

is the cumulative portfolio's return obtained from the allocation period $h-1$, with $h \in \{2, \ldots, t_{AP}\}$, and the notation (EW) is used to refer to the EW portfolio allocation model;

2. Compute the cumulative portfolio's return at time step $t+H$, obtained from the allocation made at time step t while using the portfolio allocation model selected by the chosen MAB algorithm,

$$R_{t+H}^{p.a.} = \prod_{i=1}^{H} \left(1 + R_{t+i}^{p} \right) - 1, \tag{4.11}$$

with the value corresponding to the non-normalised reward;

3. Compute the normalised reward via the following linear mapping,

$$x_{i,t} = \begin{cases} 0 & \text{if } R_{t+H}^{p.a.} \leq q_t^{\text{lo}}, \\ \frac{R_{t+H}^{p.a.} - q_t^{\text{lo}}}{q_t^{\text{hi}} - q_t^{\text{lo}}} & \text{if } q_t^{\text{lo}} < R_{t+H}^{p.a.} < q_t^{\text{hi}}, \\ 1 & \text{if } R_{t+H}^{p.a.} \geq q_t^{\text{hi}}, \end{cases} \tag{4.12}$$

with i corresponding to the action, i.e., portfolio allocation model, selected by the MAB algorithm for this time step, and q_t^{lo} and q_t^{hi} to the lower (*20*-th percentile) and upper (*80*-th percentile) sample quantiles of the set $\mathcal{R}_t^{(EW)}$ [22].

Furthermore, if $t \in \{L, \ldots, L+H-1\}$, with $(t-L) \bmod H = 0$ and $T_{AP} \leq 1$, then the attributed reward value at time step t is zero. This results from the fact that since no allocation period has occurred before the one at this time step, the portfolio return obtained via (4.11) can not be normalised as the set $\mathcal{R}_t^{(EW)}$ is still empty, and as such, in order not to benefit any of the actions, the reward attributed is zero. In addition, if $t = T-H$, with $(t-L) \bmod H = 0$ and $T_{AP} \geq 1$, then the reward is not computed, and the MAB is not updated as this allocation period was the last time the MAB algorithm had to select an action (or portfolio allocation model) since, for $t = T$, there is no computation for the portfolio's weights. At last, if $t \in \{T-H+1, \ldots, T-1\}$ and $T_{AP} > 2$, the reward is not computed, nor is the MAB updated since there would be missing observations for the portfolio returns to compute its cumulative value according to (4.11).

4.3 Performance Metrics

In order to analyse the portfolio's performance, a selection of performance metrics will be used, which have been selected to evaluate the portfolio in different categories. As such, Section 4.3.1 presents the metrics that analyse the performance of a portfolio relative to its return and risk generated. On the other hand, since the defined portfolio strategies are risk-based, evaluating them only in terms of their returns is insufficient since these were not considered initially by their objective functions [35]. Therefore, Section 4.3.2 introduces the metrics that evaluate portfolio strategies regarding their risk diversification.

4.3.1 Risk and Return Metrics

Average Return The investor's average return is given by,

$$\mathrm{AR} = \frac{1}{T - L - 1} \sum_{t=L+1}^{T} R_t, \tag{4.13}$$

where, $T - L - 1$ is the total number of out-of-sample observations.

Volatility The investor's returns volatility, or standard deviation, is computed as follows,

$$\mathrm{VO} = \sqrt{\frac{\sum_{t=L+1}^{T} (R_t - AR)^2}{T - L - 1}}. \tag{4.14}$$

Sharpe Ratio The *Sharpe ratio* measures the reward-to-risk ratio of a portfolio strategy, and is obtained by subtracting from the investor's average return the average risk-free return, with the result being divided by the investor's returns volatility.

However, in Section 4.2.2, the investor's returns defined in (4.8) already take into account the risk-free return. Therefore, its value does not need to be subtracted from the average investor's returns. Thus, given $T - L - 1$ out-of-sample observations, the Sharpe ratio is given by,

$$\mathrm{SR} = \frac{AR}{VO}. \tag{4.15}$$

Maximum Drawdown For a given time interval, the *drawdown* gives the percentage value of the difference between the maximum peak of the investor's wealth and that which she presents at the final moment of the interval. Thus, the drawdown at time step $t \in \{L+1, \ldots, T\}$ is given by,

$$DD_t = \frac{W_t}{\max_{t' \in \{L+1, \ldots, t\}} W_{t'}} - 1. \tag{4.16}$$

Thus, the *maximum drawdown* is the largest drop from a peak over the out-of-sample period $T - L - 1$, which is formally defined as,

$$\text{MDD} = \min\left(0; \min_{t \in \{L+1;...;T\}} DD_t\right): \tag{4.17}$$

4.3.2 Risk Diversification

Turnover The average proportion of wealth traded at each time step, *turnover*, is defined as the average of the absolute value of the trades across the N assets that constitute the portfolio over $T - L - 2$ out-of-sample observations (for portfolio weight vectors), with its value given by the following expression [14],

$$\text{TRN} = \frac{1}{T - L - 2} \sum_{t=L+1}^{T-1} \sum_{i=1}^{N} \|w_{i;t} - w_{i;t-1^+}\|_1: \tag{4.18}$$

Effective Number of Constituents Being a weight-based measure[5] of portfolio concentration, the *effective number of constituents* provides a quantitative estimate of weight concentration in a portfolio [13].

More formally, it is defined as the exponential of the entropy of the the portfolio weight vector's distribution. Then, for a time step $t \in \{L;...;T-1\}$, its value is given by,

$$ENC_t = \exp\left(-\sum_{i=1}^{N} w_{i;t} \ln(w_{i;t})\right): \tag{4.19}$$

Then, given $T - L - 1$ out-of-sample observations (for portfolio weight vectors), the average effective number of constituents is computed as follows,

$$\text{ENC} = \frac{1}{T - L - 1} \sum_{t=L}^{T-1} ENC_t: \tag{4.20}$$

The ENC value reaches a minimum of 1 when the portfolio is fully concentrated in a single asset, and a maximum of N when the portfolio weights are uniformly distributed. Despite its intuitive nature, this measure has no consideration for the volatility and correlation structure of the N assets that constitute the portfolio.

Effective Number of Correlated Bets Taking the limitations of the effective number of constituents metric into consideration, by being a risk-based measure of concentration, the *effective number of correlated bets*, besides taking into account the weight of each portfolio constituent, also incorporates information regarding their volatility and correlation structure [13].

More formally, it is defined as the exponential of entropy, given the distribution of the correlated constituents' contribution to the total portfolio's variance. Then, for a time step $t \in \{L;...;T-1\}$, its

[5] Weight-based measures of portfolio concentration are purely based on the analysis of the portfolio weight distribution, where the risk characteristics of the assets present in the portfolio are not considered.

value is given by,

$$ENCB_t = \exp\left(-\sum_{i=1}^{N} c_{i;t}\ln(c_{i;t})\right); \qquad (4.21)$$

where,

$$c_{i;t} = \frac{w_{i;t}(\Sigma_t \mathbf{w}_t)_i}{\mathbf{w}_t^{\top}\Sigma_t \mathbf{w}_t}; \qquad (4.22)$$

is the contribution of the i-th constituent to the portfolio's variance at time t.

Then, given $T - L - 1$ out-of-sample observations (for portfolio weight vectors and estimated covariance matrices), the average effective number of correlated bets is computed as follows,

$$\text{ENCB} = \frac{1}{T - L - 1}\sum_{t=L}^{T-1} ENCB_t. \qquad (4.23)$$

The ENCB has a minimum value of 1 if the risk of the portfolio is fully concentrated on one asset. On the other hand, it has a maximum value of N if the risk is spread evenly across all the constituents [42].

Chapter 5

Results and Discussion

This chapter provides the performance results obtained after testing the various portfolio strategies in real-world market data. The American market was selected as the environment where these would be thoroughly tested since it has a considerable influence on the global financial market behaviour. Thus, the *S&P 500* index[1] was chosen to represent it, with the choice being made according to the asset's relevance that it is composed of (with these being some of the most prominent American publicly traded companies by market capitalisation), alongside the large diversity of sectors it represents.

Moreover, the assets that constitute the portfolio[2] are precisely the same ones that belong to the S&P 500 index. However, to become eligible, the assets must have been traded between December 31, 2001, and December 31, 2021, as this was the period selected to perform the performance analysis of the various portfolio strategies. In addition, the risk-free asset used for computing the investor's excess returns corresponds to the *3-Month US Treasury Bill*. Since its daily values are the annualised returns, these had to be transformed into daily returns, which was done by dividing each daily value by 252 (average number of trading days in a year)[3]. On Table 5.1 one may find a summary of relevant information for the portfolio being traded in the American market.

Table 5.1: Descriptive summary for the American market, which includes the representative market index, the number of assets N that constitute the portfolio, the risk-free asset considered for the particular market, the time period during which the historical returns were collected and the frequency in which these are.

Market	Index	N	Risk-Free Asset	Time Period	Frequency
American	S&P 500	391	3-M US Treasury Bill	2001-12-31 → 2021-12-31	Daily

Furthermore, having a portfolio constituted of 391 assets, one has to compute at least 76440 distinct parameters during the estimation of the covariance matrix. If the data being used has a monthly frequency (240 months total) for the time period under consideration, then it would result in a number of degrees of freedom of $DF_{391,240} < 2$ per estimated parameter, computed through the following expression,

$$DF_{N,T} = \frac{NT}{2N^2}. \tag{5.1}$$

Consequently, by having a low number of degrees of freedom, the various components of the covariance matrix are estimated with less precision, being, therefore, more error-prone. Then, to overcome this issue, it was decided to use higher frequency data, particularly daily values. This action led to an increase in

[1] In the S&P 500 index, the weight assigned to an asset is determined according to the total market values of its outstanding shares, thus being called a *Capitalised Weighted* index.

[2] On Table B.1 one may find more information regarding the assets that constitute the portfolio on the American market.

[3] The 3-Month US Treasury Bill's daily annualised returns were taken from https://fred.stlouisfed.org/.

the number of degrees of freedom by twelve-fold, thus helping to reduce the sampling error [25].

To measure the statistical significance of the difference between the Sharpe ratios obtained from two portfolio strategies, the p-value between the two is computed. In particular, to compute the p-value for the Sharpe ratios, the bootstrap methodology developed by Ledoit and Wolf [31] is used, which was designed purposely for when the portfolio returns have fat tails and are of a time series nature (as the portfolio returns are serially correlated). Then, considering two portfolio strategies i and j, with $\mu_i; \mu_j$ and $\sigma_i; \sigma_j$ as their true mean and volatility, respectively, the null hypothesis in which the Sharpe ratio of portfolio strategy i is equal to that of portfolio j is given by,

$$H_0 : \frac{\mu_i}{\sigma_i} = \frac{\mu_j}{\sigma_j}; \tag{5.2}$$

whereas the alternative hypothesis is given as,

$$H_1 : \frac{\mu_i}{\sigma_i} \neq \frac{\mu_j}{\sigma_j}: \tag{5.3}$$

To test which hypothesis is valid, a two-sided p-value is computed while using a studentized circular block bootstrap proposed in Ledoit and Wolf [31], with $B = 5000$ bootstrap re-samples and a block size equal to $b = 5$.

Moreover, an additional portfolio strategy is used during this analysis called CHANCE. Alongside EXP3 and EXP3.S, this one also selects a portfolio allocation model (MV, ERC or MD) according to a probability distribution at every allocation period. However, no reward is needed after each selection phase, as it is assumed that this strategy uses a uniform distribution to select one of the three possible actions, which remains unchanged over time. The purpose of using CHANCE is to allow both EXP3 and EXP3.S to be compared against a less complex method of selecting a portfolio allocation model at each allocation period.

Additionally, given that CHANCE, EXP3 and EXP3.S select a particular action according to a probability distribution, during their computation, the outcome is dependent on the seed being used, with this one determining how a random value is generated. Thus, to have reliable results that do not depend on the seed value, these three portfolio strategies were computed using 40 different seeds. After that, the returns and wealth at each time step were averaged between the 40 different values. On the other hand, the metrics that depended on the portfolio's weights (all risk diversification metrics) were individually computed for each seed, and only after was the average computed between the 40 different seeds.

5.1 Standard Results

Initially, the data is analysed according to a set of parameters considered the "standard" ones, Table 5.2. More precisely, the value of the parameter L is selected by considering the number of assets that constitute the portfolio. Given that the American market portfolio has a total number of 391 assets, the value of L should be at least or higher than this value in order to avoid errors during the estimation of the covariance matrix. Therefore, after testing different values for this parameter, $L = 500$ proved to be feasible[4]. Furthermore, for parameter H, it was assigned a value of 1 in order to allow EXP3 and EXP3.S to have a significant number of observations from which to learn. In addition, the various portfolio strategies are not subject to turnover restrictions, and as a result, the value of TO is set to 0. At last, for the CORE+ only parameters, besides K (which has a fixed value of 3), these were set deterministically as previously described in Section 3.2.

Table 5.2: Standard parameter values for the CORE and CORE+ formulations.

L	H	TO	K	(EXP3)	(EXP3:S)	
500	1	0	3	0.0206	0.0793	0.0002

Thus, given that the window size considered presents a value of $L = 500$, the first allocation period takes place on December 23, 2003. As a result, since the reallocation is performed daily, this gives a total of 4538 out-of-sample observations. Additionally, the returns are only obtained at the subsequent time step following the first allocation period. At last, it is considered that the investor's initial wealth is 1.

Moreover, Table 5.3 shows the out-of-sample results for the risk and return metrics when $TC = 0$. In parentheses, below the value obtained by each portfolio strategy for the Sharpe ratio metric, the p-values obtained for the hypothesis test for the Sharpe ratio difference between the respective portfolio strategy and EW are given (the latter having been selected based on the fact that it is considered in the literature as a *benchmark* portfolio allocation model [14]). Additionally, in bold, the best value obtained for a given metric from all portfolio strategies is represented, and underlined, the best value obtained for a given metric between CHANCE, EXP3 and EXP3.S is given. Then, from the results obtained in this table, the following remarks can be made:

- The MD portfolio presents the highest average return and Sharpe ratio, and between CHANCE and the two MAB strategies, it should be noted that EXP3.S is the one that offers the best performance by being superior in both metrics. However, its average return is lower than that obtained by ERC and MD while still being higher than that of MV. On the other hand, the EXP3.S presents a higher Sharpe ratio than ERC and MV but lower than the one obtained by MD;

- According to the volatility results, MV achieves the lowest value while EW has the highest one. As for CHANCE and the two MAB strategies, the former achieves the lowest value, but not by a significant margin;

- As for the p-value results, MD's Sharpe ratio is statistically different from EW's, as the null hypothesis is rejected at a significance level of 1%. Additionally, the EXP3.S's Sharpe ratio is also

[4] The values 750, 1000 and 2000 were also tested for the L parameter. However, these did not show significant improvements in the performance of the portfolio allocation models. As such, $L = 500$ was the best option since it permitted to have more out-of-sample observations.

statistically di erent from the EW's, with the null hypothesis being rejected at a signi cance level of 10%;

- In terms of maximum drawdown, MV achieves the lowest value while EW has the highest one. Additionally, between CHANCE and the two MAB strategies, the former presents the lowest value among the three, while EXP3.S presents the highest.

Table 5.3: Results of the risk and return metrics obtained by each portfolio strategy for the set of standard parameters and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
AR (%)	0.0555	0.0342	0.0505	**0.0634**	0.0477	0.0491	0.0504
VO (%)	1.2848	**0.7965**	1.1566	1.0152	0.9509	0.9540	0.9617
SR	0.0432	0.0429	0.0437	**0.0625**	0.0502	0.0515	0.0524
	(1.00)	(0.98)	(0.72)	(0.01)	(0.17)	(0.11)	(0.06)
MDD (%)	-52.3988	**-40.0576**	-50.0698	-50.3980	-44.9109	-45.5458	-45.4339

Moreover, from the results obtained in the risk diversi cation metrics presented in Table 5.4, the following remarks can be made:

- Among all the portfolio strategies, EW achieved the lowest turnover value. As for CHANCE and the two MAB strategies, EXP3.S presented the lowest turnover value of the three. Also, it should be noted that CHANCE has the highest turnover among all portfolio strategies;

- The EW portfolio presents the highest ENC value, with an ENC of 391 (which comes from the fact that the weights are distributed evenly between all assets). Between CHANCE and the two MAB strategies, EXP3.S is the one that has the highest ENC;

- The ERC portfolio presents the highest ENCB value, with an ENCB of 391 (which comes from the fact that the risk is distributed equally across all assets). Additionally, between CHANCE and the two MAB strategies, EXP3.S has the highest ENCB value.

Table 5.4: Results of the risk diversi cation metrics obtained by each portfolio strategy for the set of standard parameters and $TC = 0$.

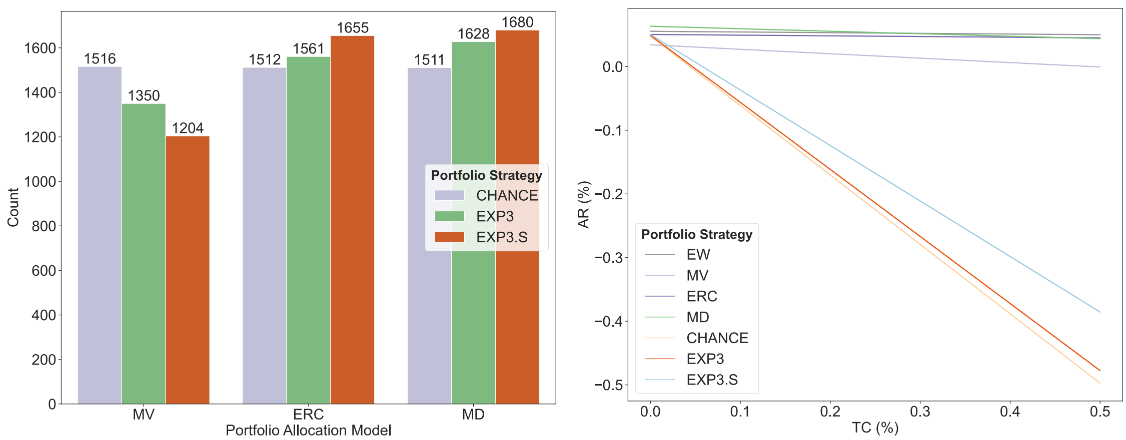
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
TRN (%)	**1.0475**	6.9413	1.0592	3.8889	108.9510	105.2874	87.1554
ENC	**391.0000**	21.3923	368.0266	28.1436	139.0921	142.9381	150.3059
ENCB	367.9986	21.4984	**391.0000**	28.8901	147.0282	151.1246	158.9318

Having said that, MD presents the best results in terms of average return and Sharpe ratio. However, when it comes to risk diversi cation, this one could not achieve the same good results, which is particularly clear by the values that it achieved on the ENC and ENCB metrics. On the other hand, MV stands out in terms of volatility and maximum drawdown, obtaining the best values among the various portfolio strategies.

As for CHANCE and the two MAB strategies, among the three, EXP3.S achieved superior results in terms of average returns and Sharpe ratio, which can be explained by the fact that it has selected MD more often than ERC and MD, as given in Figure 5.1(a). On the other hand, since EXP3.S selected MV less often, the results achieved in volatility and maximum drawdown were the worst among the three strategies.

The reason for EXP3 and EXP3.S having selected MV less often than MD is a result of the reward function being used. As MD achieves the best returns on average, the reward function should output a higher reward when this portfolio allocation model is selected than when MV is, resulting in the latter having lower average returns than the former. Consequently, over time, the probability of selecting MD increases while the MV one decreases, which explains the difference in the number of times each was selected.

Moreover, between EXP3 and EXP3.S, the latter was able to learn faster which of the three actions was giving higher returns, as it selected MD more often than EXP3 did. As a result, EXP3.S achieved higher average returns than EXP3. Additionally, as EXP3.S selected more often ERC than CHANCE and EXP3 did, it benefited from the ERC's risk diversification results since it presented better values in terms of ENC, ENCB and turnover than CHANCE and EXP3 were able to achieve.



(a) $H = 1$, $TO = 0$ and $TC = 0$

(b) $H = 1$, $TO = 0$ and $TC \in [0, 0.005]$

Figure 5.1: (a) Number of times CHANCE, EXP3 and EXP3.S selected each of the available actions (MV, ERC and MD). (b) Average returns obtained by EW, MV, ERC, MD, CHANCE, EXP3 and EXP3.S for a set of positive TC values.

The CHANCE, EXP3 and EXP3.S portfolio strategies have exceptionally high turnover values, which could ultimately impact their return when proportional transaction costs are different from zero. Thus, in order to analyse their impact, Table 5.5 presents the average returns of the various portfolio strategies when subjected to different TC values. From the results contained in this table, it is possible to make the following remarks:

- Both EW and ERC, despite having lower average returns than MD when $TC = 0$, since these have a lower turnover than MD, as the value of TC increases, the average return of the two former portfolio strategies end up meeting those achieved by MD from which point forward these start to gain higher average returns than the latter one;
- As for the two MAB strategies, these were only able to achieve positive average returns up to $TC = 0.0004$, with the decrease in average returns being more noticeable in EXP3 than in EXP3.S, as the former has a higher turnover. Nevertheless, both portfolio strategies have continuously better average returns than CHANCE with the increase in TC value, which is due to the fact that CHANCE presents the highest turnover among the three.

Table 5.5: Average returns of each portfolio strategy in the American market for di erent TC values, $H = 1$ and $TO = 0$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0555	0.0342	0.0505	0.0634	0.0477	0.0491	0.0504
0.02	0.0552	0.0328	0.0503	0.0626	0.0259	0.0281	0.0329
0.04	0.0550	0.0314	0.0501	0.0619	0.0041	0.0070	0.0155
0.06	0.0548	0.0300	0.0499	0.0611	-0.0177	-0.0141	-0.0019
0.08	0.0546	0.0286	0.0496	0.0603	-0.0395	-0.0351	-0.0194
0.10	0.0544	0.0272	0.0494	0.0595	-0.0613	-0.0562	-0.0368
0.15	0.0539	0.0237	0.0489	0.0575	-0.1158	-0.1089	-0.0804
0.20	0.0533	0.0202	0.0483	0.0556	-0.1703	-0.1616	-0.1240
0.25	0.0528	0.0168	0.0478	0.0536	-0.2248	-0.2142	-0.1676
0.30	0.0522	0.0133	0.0473	0.0517	-0.2793	-0.2669	-0.2112
0.35	0.0517	0.0098	0.0467	0.0497	-0.3338	-0.3196	-0.2548
0.40	0.0512	0.0063	0.0462	0.0478	-0.3883	-0.3722	-0.2984
0.45	0.0506	0.0028	0.0456	0.0458	-0.4428	-0.4249	-0.3420
0.50	0.0501	-0.0006	0.0451	0.0439	-0.4973	-0.4776	-0.3856

It can be concluded that proportional transaction costs present a substantial problem in the returns obtained from a given portfolio strategy, with its impact being directly related to the turnover of each portfolio strategy, as the higher the turnover is, the higher the impact of proportional transaction costs on returns. This problem is even more critical for EXP3, EXP3.S and CHANCE, as these become infeasible for TC values above 0.0004. In addition, for TC values above 0, their average returns are consistently lower than any portfolio allocation model tested, Figure 5.1(b).

In contrast, if TC has a value of 0, both EXP3 and EXP3.S can be used to select the \optimal" portfolio allocation model at each allocation period. The reasoning for coming up with this conclusion is due to the fact that both achieved a higher Sharpe ratio than EW, MV and ERC, only falling behind the one obtained by MD. Additionally, on the other three risk and return metrics, the values obtained are often superior to the average of those obtained by MV, ERC, and MD combined. Within the risk diversi cation metrics, these were able to have better ENC and ENCB values than MV and MD, only having inferior numbers to those obtained by EW and ERC. Overall, in most performance metrics, EXP3.S obtained better results than EXP3, only falling behind (but not signi cantly) in terms of volatility and maximum drawdown. The superior performance of EXP3.S was mainly due to the noise added during the update of the weight of the selected action, which took into account the non-stationarity of the best action.

5.2 Solutions to Decrease the Turnover

In order for the usage of EXP3 and EXP3.S as portfolio strategies to become feasible in scenarios where proportional transaction costs are di erent from zero, the turnover that these two generate must become lower than the one it had while using the standard parameters. By decreasing the turnover, it will allow the portfolio strategies to pay a smaller amount in proportional transaction costs, thus leading to higher average returns than those obtained previously for TC values greater than zero.

This section proposes two di erent solutions to decrease the turnover, for which the obtained results are analysed. The rst solution consists of increasing the allocation period H, Section 5.2.1, while the second on adding a restriction to the portfolio turnover TO, as in (4.3), during the optimisation of a given portfolio allocation model, Section 5.2.2.

5.2.1 Solution 1: $H > 1$

The rst solution consists of increasing the duration taken between two portfolio allocation periods. Introducing this solution may reduce turnover since the update of the portfolio weights is performed less often, remaining unchanged over a longer period.

As a result, two values were tested, $H = 5$ and $H = 20$, which correspond to performing the portfolio allocation on a weekly and monthly basis, respectively. For these new H values, all the parameters used by the CORE+ formulation, except K, su er changes since the total number of allocation periods T_{AP} is di erent (in this particular case, the T_{AP} is lower than it was when $H = 1$). Table 5.6 shows the values assigned to the backtesting parameters for the two H values tested.

Table 5.6: Parameter values for the CORE and CORE+ formulations when H takes values of 5 and 20.

L	H	TO	K	(EXP3)	(EXP3:S)	
500	5	0	3	0.0460	0.1617	0.0011
500	20	0	3	0.0919	0.2936	0.0044

Table 5.7 shows the values of the risk and return metrics for $H = 5$ and $H = 20$, while having $TC = 0$. The following remarks can be made:

- As the value of H increases, the Sharpe ratio of MV, ERC and MD decreases. As for the volatility, it increases in both MV and MD, while for ERC, it decreases. As for the maximum drawdown, MV achieves better results with the H value increment, whereas ERC and MD are negatively impacted;

- For $H = 5$, the average returns and Sharpe ratio for EXP3 and EXP3.S were positively impacted, with the latter showing superior results on both metrics than the former. On the other hand, for $H = 20$, the results obtained on the two previously mentioned metrics were not as good as those obtained for $H = 1$ (and $H = 5$). Additionally, for each value of H, CHANCE always achieved inferior average returns and Sharpe ratio than the two MAB strategies;

- At last, both CHANCE and the MAB strategies had a Sharpe ratio statistically different from EW's for $H = 5$, with the null hypothesis being rejected at a significance level of 5% for the two MAB strategies and 10% for CHANCE. On the other hand, for $H = 20$, only the MAB strategies could have their Sharpe ratio statistically different from EW's, with the null hypothesis being rejected at a significance level of 10%.

Table 5.7: Results of the risk and return metrics obtained by each portfolio strategy for $H \in \{5, 20\}$, $TO = 0$ and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
$H = 5$							
AR (%)	0.0547	0.0342	0.0499	**0.0628**	0.0500	0.0512	0.0519
VO (%)	1.2812	**0.8091**	1.1546	1.0199	0.9549	0.9620	0.9658
SR	0.0427	0.0423	0.0432	**0.0615**	0.0524	0.0532	0.0537
	(1.00)	(0.97)	(0.72)	(0.01)	(0.06)	(0.04)	(0.03)
MDD (%)	-53.6305	**-39.1593**	-51.0484	-51.3130	-44.3418	-44.8166	-45.0579
$H = 20$							
AR (%)	0.0536	0.0332	0.0491	**0.0609**	0.0477	0.0492	0.0492
VO (%)	1.2763	**0.8144**	1.1524	1.0266	0.9544	0.9784	0.9805
SR	0.0420	0.0407	0.0426	**0.0594**	0.0499	0.0503	0.0502
	(1.00)	(0.89)	(0.69)	(0.01)	(0.12)	(0.09)	(0.09)
MDD (%)	-53.7687	**-38.8964**	-51.2317	-51.4007	-45.4391	-45.7664	-46.4559

Table 5.8 shows the values of the risk diversification metrics for $H = 5$ and $H = 20$, while $TC = 0$. The following observations can be made:

- The increment in the value of H positively impacted the turnover generated by each portfolio strategy, with the best results being observed for $H = 20$. Between the two MAB strategies, EXP3.S stood out in the turnover value that it yielded with the increment of the H value. More specifically, for $H = 5$ and $H = 20$, it generated a value of 18.37% and 4.83%, respectively, whereas, for $H = 1$, it yielded a turnover value of 87.15%;

- As for the ENC and ENCB, no significant changes have occurred to the values achieved by MV, ERC and MD. On the other hand, for EXP3 and EXP3.S, the value for these two metrics only improved with $H = 20$, whereas for $H = 5$, they were inferior to those obtained for $H = 1$;

- At last, the MAB strategies had better results on all three metrics than CHANCE for the two H values tested.

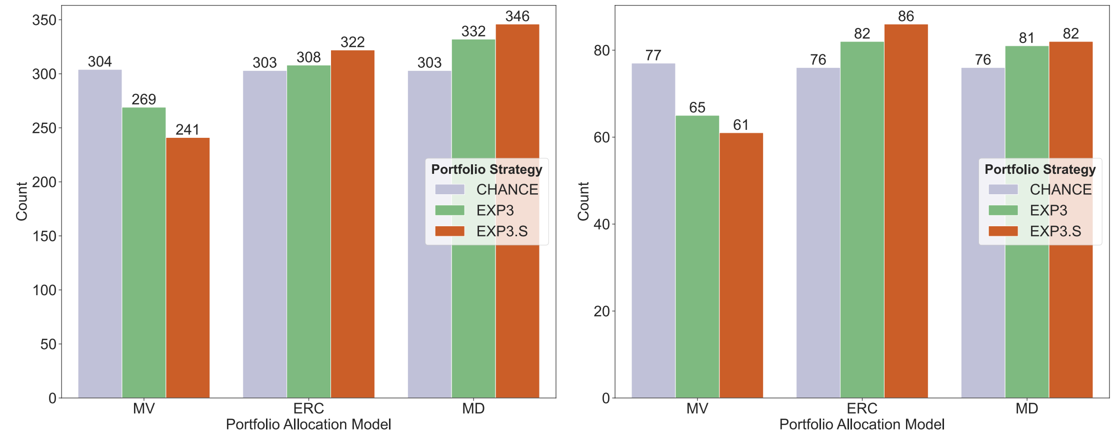
Table 5.8: Results of the risk diversification metrics obtained by each portfolio strategy for $H \in \{5; 20\}$, $TO = 0$ and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
$H = 5$							
TRN (%)	**0.4809**	2.7155	0.5013	2.0306	22.0929	21.4117	18.3665
ENC	**390.8926**	21.7233	367.9441	28.1498	139.3118	141.3463	146.9212
ENCB	367.8630	21.8433	**390.8669**	28.8632	147.2084	149.3993	155.2784
$H = 20$							
TRN (%)	**0.2445**	1.3124	0.2623	1.1619	5.7242	5.3937	4.8315
ENC	**390.5364**	21.7110	367.7031	28.1494	138.7399	148.2132	153.9011
ENCB	367.3231	21.8261	**390.3831**	28.7490	146.4591	156.5746	162.6067

From the results obtained above, it can be concluded that, despite having a decrease in Sharpe ratio, the increment of H leads to a turnover decrease for MV, ERC and MD. Therefore, in a market where proportional transaction costs are taken into account, the Sharpe ratio will not be impacted as much for a higher H value as it is for $H = 1$. The decrease in turnover leads to lower spending on proportional transaction costs and, consequently, to an increase in the Sharpe ratio compared to what would have been achieved by a lower H value with it being a higher turnover. As for the two MAB strategies, most metrics were positively impacted by the increase of the H value. For example, for $H = 5$, the most significant improvement was in the increase of the Sharpe ratio, whereas for $H = 20$, the improvement was particularly notorious in the overall risk diversification metrics.

Furthermore, according to Figure 5.2(a), it can be observed that EXP3 and EXP3.S continue to have an accurate learning process by selecting more often the best action, MD, and less often the worst action, MV. In addition, in Figure 5.2(b), it is noticeable that, for $H = 20$, both portfolio strategies selected ERC more often than MD, which led to a decrease in average returns and Sharpe ratio, as observed previously. The fact that both strategies did not select the most the best action, MD, may be explained by the small number of observations that these two strategies have at their disposal to learn which portfolio allocation model (MV, ERC or MD) is the "optimal" one. A solution to this problem would be to increase the learning rate parameter in both strategies ($^{(EXP3)}$ and $^{(EXP3.S)}$).

Moreover, in Figure 5.3, the average returns gained by CHANCE, EXP3 and EXP3.S with the increase of the proportional transaction cost for the three H values tested are represented. From the results it presents, it is possible to verify an improvement in the average returns yielded by CHANCE, EXP3 and EXP3.S as the value of H increases. It is also noticeable that EXP3.S presents higher average returns than the other two strategies for any of the three H values for $TC > 0$. Additionally, in Figure 5.3(b), it is represented the average returns over the increase of proportional transaction costs for all portfolio strategies while having $H = 20$. This one shows that despite the increase in the average return by the MAB strategies, these continue to show lower values than those obtained by EW, ERC and MD with the increase of TC. Despite this result, EXP3.S is able to achieve higher returns than MV up until a TC

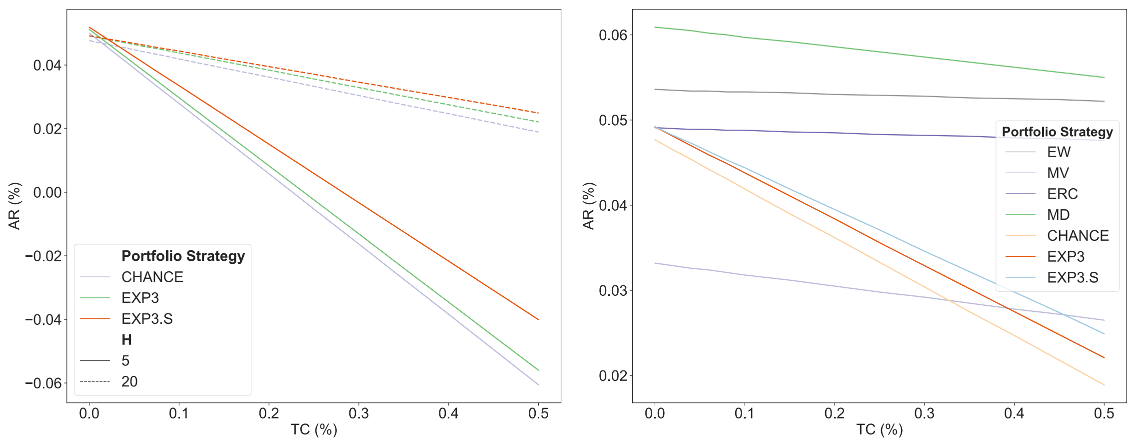


(a) $H = 5$

(b) $H = 20$

Figure 5.2: Number of times CHANCE, EXP3 and EXP3.S selected each of the available actions (MV, ERC and MD) for $H \in \{5, 20\}$, $TO = 0$ and $TC = 0$.

value of around 0.005[5].



(a) Values obtained by CHANCE, EXP3 and EXP3.S for $H \in \{5, 20\}$ and $TO = 0$.

(b) Values obtained by EW, MV, ERC, MD, CHANCE, EXP3 and EXP3.S for $H = 20$ and $TO = 0$.

Figure 5.3: Average returns obtained by using solution 1 for $TC \in [0, 0.005]$.

At last, from a global point of view, according to the results obtained by this solution to decrease the turnover, all portfolio strategies benefited from the increase of the H value when proportional transaction costs are considered. Between CHANCE and the two MAB strategies, since EXP3.S achieved the lowest turnover for all three H values tested, it outperformed CHANCE and EXP3 in terms of average returns

[5] In Tables A.1 and A.2 the average returns obtained for the TC values tested for the H values of 5 and 20 are represented, respectively

when subjected to proportional transaction costs. Besides that, EXP3 also has higher ENC and ENCB compared to MV, MD, CHANCE and EXP3, demonstrating good performance in risk diversification.

Thus, despite both MAB strategies achieving a better Sharpe ratio for $H = 5$, the turnover generated is far from optimal if proportional transaction costs are considered. As such, $H = 20$ stands out as a better solution for reducing the turnover of the MAB strategies without compromising their learning capability and being superior to CHANCE. Its superiority is evident when the average returns from the three H values tested are compared when the TC value increases and the performance achieved in risk diversification, particularly in both ENC and ENCB metrics.

5.2.2 Solution 2: $TO > 0$

As previously analysed, the decrease in turnover leads to lower expenditure in proportional transaction costs. Therefore, the second solution focuses precisely on decreasing the turnover by restricting its value during the portfolio optimisation at each allocation period, with the restriction being given in (4.3). Then, four different TO values were tested: 0.20, 0.15, 0.10 and 0.05. The values for the remaining parameters are the same as the ones presented in Table 5.2.

Table 5.9 shows the values of the risk and return metrics for the proposed TO values, while $TC = 0$. The following remarks can be made:

- The restriction in the turnover led to an increase in volatility in both MV and MD, which increases with the decrease of the TO value. As for ERC, the volatility remained stable despite the restriction's addition. Additionally, although for $TO = 0.20$ the average returns decreased on both MV and MD, for lower TO values, the average returns were superior to those obtained before adding the restriction. As for ERC, this one suffered no changes in the generated average returns. Regarding the Sharpe ratio, for the TO values tested, MV could never yield better results than before the restriction was added. As for MD, for $TO \leq 0.10$, its Sharpe ratio and maximum drawdown tended to increase in value as TO diminished. Since ERC had no significant changes in volatility and average returns, its Sharpe ratio also remained unchanged. At last, with the TO restriction, MV always had better maximum drawdown results, whereas ERC's remained unchanged;
- For CHANCE and the two MAB strategies, their average returns and Sharpe ratio were initially negatively impacted by the insertion of the turnover restriction during the optimisation phase. However, as the TO value decreased, these values surpassed those obtained when the restriction was not in use. For example, for $TO \leq 0.10$, the three portfolio strategies started to yield superior average returns, and for $TO = 0.05$, their Sharpe ratio also turned superior. As for the volatility and maximum drawdown, on no occasion were these strategies able to achieve lower values than those obtained before adding the restriction;
- Regarding EW, its values remained unchanged for all TO values tested as the weights that this

portfolio strategy attributes for each period are always the same, for which no weight optimisation is used.

- Between MV, ERC and MD, only the latter was able to reject the null hypothesis that its Sharpe ratio is statistically di erent from EW's at a 5% signi cance level, which it did across all TO values tested. As for CHANCE and the two MAB strategies, these were only able to reject the null hypothesis for $TO = 0.05$, with the rst rejecting it at a 10% signi cance level, while the two MAB strategies rejected it at a 5% one.

Table 5.9: Results of the risk and return metrics obtained by each portfolio strategy for $TO \in \{0.20; 0.15; 0.10; 0.05\}$, $H = 1$ and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
$TO = 0.20$							
AR (%)	0.0555	0.0317	0.0505	**0.0613**	0.0410	0.0428	0.0460
VO (%)	1.2848	**0.8358**	1.1558	1.0367	0.9669	0.9646	0.9745
SR	0.0432	0.0380	0.0437	**0.0591**	0.0424	0.0444	0.0472
	(1.00)	(0.54)	(0.71)	(0.02)	(0.90)	(0.83)	(0.40)
MDD (%)	-52.3988	**-39.4968**	-50.0747	-51.6569	-45.7544	-45.7308	-45.9143
$TO = 0.15$							
AR (%)	0.0555	0.0326	0.0505	**0.0671**	0.0445	0.0458	0.0483
VO (%)	1.2848	**0.8415**	1.1556	1.0772	0.9765	0.9734	0.9777
SR	0.0432	0.0388	0.0437	**0.0623**	0.0456	0.0471	0.0494
	(1.00)	(0.62)	(0.69)	(0.01)	(0.67)	(0.49)	(0.22)
MDD (%)	-52.3988	**-38.7399**	-50.0960	-51.7374	-45.3542	-45.2559	-45.4158
$TO = 0.10$							
AR (%)	0.0555	0.0332	0.0505	**0.0777**	0.0510	0.0519	0.0533
VO (%)	1.2848	**0.8526**	1.1557	1.2258	1.0210	1.0202	1.0344
SR	0.0432	0.0389	0.0437	**0.0634**	0.0499	0.0509	0.0515
	(1.00)	(0.62)	(0.74)	(0.02)	(0.27)	(0.20)	(0.12)
MDD (%)	-52.3988	**-38.6485**	-50.1417	-59.2374	-47.2599	-47.0926	-48.4936
$TO = 0.05$							
AR (%)	0.0555	0.0359	0.0503	**0.0961**	0.0655	0.0668	0.0674
VO (%)	1.2848	**0.8838**	1.1559	1.4582	1.1599	1.1824	1.2004
SR	0.0432	0.0406	0.0435	**0.0659**	0.0565	0.0565	0.0561
	(1.00)	(0.75)	(0.82)	(0.02)	(0.06)	(0.05)	(0.04)
MDD (%)	-52.3988	**-39.4150**	-50.3294	-60.1106	-53.5578	-53.9341	-53.8977

Table 5.10 shows the values of the risk diversi cation metrics for the proposed TO values, while $TC = 0$. The following remarks can be made:

- The turnover restriction led to a decrease in the turnover generated by the portfolio strategies, with its value decreasing as TO assumed lower values. The impact was most noticeable in CHANCE and the MAB strategies, with EXP3.S achieving the lowest turnover values of the three, regardless of the TO value;

- As for ENC and ENCB, these increased for MV with the restriction's introduction, with their values assuming higher numbers as TO decreased in value. On the other hand, ERC's values did not suffer noticeable changes, and for MD, despite having had an increase for $TO = 0.20$ and $TO = 0.15$, for the other two values tested, both ENC and ENCB were inferior to those obtained before the restriction was in place. At last, for CHANCE and the MAB strategies, the restriction addition led to a significant decrease in both metrics for all three strategies. As the TO value decreased, both ENC and ENCB values never returned to those they had before the restriction. In fact, for $TO = 0.05$, where all three portfolio strategies had the best Sharpe ratio performance, these had the lowest ENCB values registered so far, and as for ENC, its value was superior for both CHANCE and EXP3 when compared with the other TO values tested, and the worst for EXP3.S.

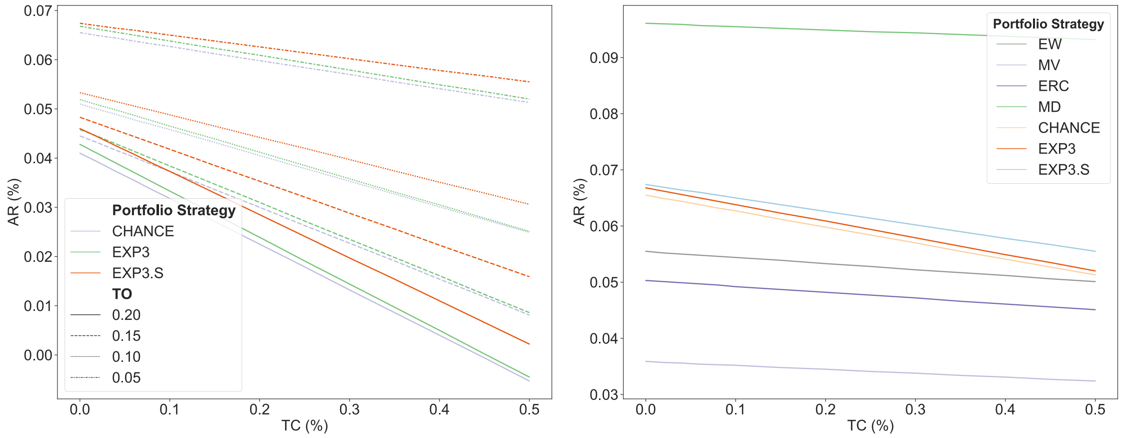
Table 5.10: Results of the risk diversification metrics obtained by each portfolio strategy for $TO \in \{0.20; 0.15; 0.10; 0.05\}$, $H = 1$ and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
$TO = 0.20$							
TRN (%)	**1.0475**	1.6701	1.0562	1.8590	9.2313	9.4333	8.7348
ENC	**391.0000**	22.3447	367.9943	29.4611	54.6629	67.3103	111.5170
ENCB	367.9986	22.2749	**390.9945**	29.6959	66.0363	80.1689	125.9920
$TO = 0.15$							
TRN (%)	**1.0475**	1.4856	1.0540	1.5237	7.2571	7.4224	6.4536
ENC	**391.0000**	23.7177	367.9734	29.0381	56.4476	69.7590	103.8900
ENCB	367.9986	23.6612	**390.9889**	26.8610	65.0972	80.0173	114.5118
$TO = 0.10$							
TRN (%)	**1.0475**	1.1369	1.0484	1.0591	5.1931	5.3402	4.5106
ENC	**391.0000**	25.7976	367.9720	27.3528	60.3122	73.2213	108.2046
ENCB	367.9986	25.6821	**390.9837**	23.8422	66.1727	80.5704	115.2299
$TO = 0.05$							
TRN (%)	1.0475	0.6612	1.0213	**0.5485**	2.8237	2.9277	2.3609
ENC	**391.0000**	29.6179	367.9219	23.3071	64.6295	79.9109	90.6367
ENCB	367.9986	27.8215	**390.9609**	16.8192	60.9568	75.9938	86.2311

Furthermore, to better understand the performance of this solution when proportional transaction costs are taken into account, the various TO values suggested were applied to CHANCE, EXP3 and EXP3.S for a set of selected TC values, as shown in Figure 5.4(a)[6].

Thus, according to the results in Figure 5.4(a), it is possible to verify that as the TC value increases, the distance in average returns from EXP3.S and the other two strategies increases, with the former presenting higher average returns of the three. Additionally, the distance between the average returns of EXP3.S relative to EXP3 and CHANCE decreases as TO tends to lower values. As for the average returns, it is noticeable that as TO decreases, the average returns increase for $TC = 0$, and as TC increases, these decrease less. Note also that only for $TO = 0.20$ and $TC = 0.005$ do CHANCE, and EXP3 show a negative average return, whilst for EXP3.S, the average returns never reach negative values

[6] The results for the remaining portfolio strategies are represented in Tables A.3, A.4, A.5 and A.6, which correspond to TO values of 0.20, 0.15, 0.10 and 0.05, respectively.



(a) Values obtained by CHANCE, EXP3 and EXP3.S for $TO \in \{0.20, 0.15, 0.10, 0.05\}$ and $H = 1$.

(b) Values obtained by EW, MV, ERC, MD, CHANCE, EXP3 and EXP3.S for $TO = 0.05$ and $H = 1$.

Figure 5.4: Average returns obtained by using solution 2 for $TC \in [0, 0.005]$.

(at least till $TC = 0.005$).

Then, of all the TO values tested, it is notable that 0.05 is the best value across all three strategies. As such, to better grasp their performance against other portfolio strategies, such as EW, MV, ERC and MD, Figure 5.4(b) represents the average returns as TC increases in value for all the portfolio strategies with $TO = 0.05$. From the represented results, it is possible to see that CHANCE, EXP3, and EXP3.S were always superior to EW, MV and ERC, regardless of the value of TC. However, their average returns were consistently lower than those obtained by MD.

Moreover, to evaluate the selection process by which CHANCE and the two MAB strategies go through, Figure 5.5 has represented the number of times each portfolio allocation model (MV, ERC and MD) is selected for $TO = 0.05$. According to the results, it is possible to verify that both MV and ERC were selected more often by EXP3 and EXP3.S than MD was[7], despite the former having lower average returns than the latter. On the other hand, when the turnover restriction was not in action, the opposite scenario happened, i.e., MD was selected more often than both MV and ERC.

Therefore, according to these results, EXP3 and EXP3.S were on no occasion able to live up to what they were designed to do. I.e., since MD presents an average return higher than MV and ERC, it is possible to verify that both EXP3 and EXP3.S have difficulty detecting which of the portfolio allocation models is "optimal". For example, for TO=0.05, there is no difference in the Sharpe ratio between CHANCE and EXP3, with EXP3.S having a lower value due to its higher volatility. Then, for this TO value, one can claim that there is no superiority between CHANCE and EXP3, as the only apparent difference between the two is the superior ENC and ENCB performance yielded by EXP3. On the other

[7] Similar results were obtained for $TO \in \{0.20, 0.15, 0.10\}$.

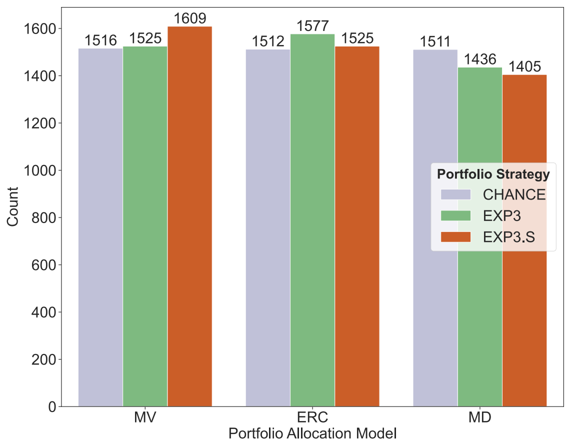


Figure 5.5: Number of times CHANCE, EXP3 and EXP3.S selected each of the available actions (MV, ERC and MD) for $TO = 0.05$, $H = 1$ and $TC = 0$.

hand, despite having a lower Sharpe ratio than the previous two, EXP3.S presents a lower turnover, higher ENC and ENCB (risk diversification), and higher average returns.

Moreover, Table 5.11 gives the TRN[8] and DIF percentage values for the three MAB strategies tested throughout this study. The DIF measure corresponds to the average value obtained per allocation period between the accumulated absolute difference of the weights generated by the portfolio allocation model selected by the MAB algorithm on each allocation period when the turnover constraint restricts the model, $w_{i,t}^{(turnover)}$, and when it does not, $w_{i,t}^{(standard)}$. More specifically, DIF is defined as follows,

$$\text{DIF} = \frac{1}{T_{AP}} \sum_{t=L}^{T-1} DIF_t \cdot 1_{\{(t-L) \bmod H = 0\}}, \tag{5.4}$$

where,

$$DIF_t = \sum_{i=1}^{N} \| w_{i,t}^{(turnover)} - w_{i,t}^{(standard)} \|_1, \quad \forall t \in \{L, \dots, T-1\}. \tag{5.5}$$

From the results obtained in Table 5.11, it is possible to verify that the average value per allocation period of the accumulated absolute difference between the weights generated by the allocation model with and without the turnover constraint is quite significant as, for example, the lowest DIF value yielded was of approximately 127% by EXP3.S. In other words, the weights generated in each allocation period via the allocation model selected by the MAB algorithm with the turnover constraint are considerably distant from those obtained without any turnover constraint.

Since the MAB algorithm presents itself with a set of three different portfolio allocation models (with each one having its individual properties) from which to select, on the occasion that it ends up selecting on an allocation period a model different from the one selected on the previous period, the

[8] These results are already available in Table 5.10. However, these are again displayed to be easier to make a one-to-one comparison with DIF.

weight di erence if no turnover restriction was in place would be quite signi cant. However, while using the turnover restriction when optimising the objective's function, the weights generated will not be too distant from those at the end of the previous time step, which is observable according to TRN values. Thus, paired with the DIF results, this re ects the lack of optimisation freedom that is brought by the turnover restriction. As such, the turnover constraint, when used side by side with the MAB algorithm's mechanism of selecting the portfolio allocation model in each allocation period, disables the portfolio allocation model from optimising the objective function freely, as its addition brings a signi cant amount of optimising limitations to the model. Therefore, under these circumstances, the theoretical guarantees brought by the MAB algorithms cannot be assured.

Thus, these results explain why EXP3 and EXP3.S have di culties when learning the \optimal" allocation model since the rewards generated from the selected model do not present a direct connection with the weights that would be originally generated without the turnover constraint. For example, assuming that in allocation period one, the MAB algorithm selected ERC as the allocation model and in allocation period two, it selects MD, according to the results above, the weights generated will be closer to those generated by ERC in the previous allocation period than those that would be generated by MD if there were no turnover constraint. Nevertheless, despite the weights being closer to ERC's, the reward is attributed to MD, thus causing problems in the overall learning process of EXP3 and EXP3.S.

Table 5.11: Results for the TRN and DIF measures yielded by CHANCE, EXP3 and EXP3.S.

	CHANCE	EXP3	EXP3.S
TRN (%)	2.8237	2.9277	2.3609
DIF (%)	140.8295	138.1260	127.4856

Overall, the solution presented here was indeed able to positively impact the decrease in the turnover generated by the various strategies, with its impact being more noticeable in the turnover reduction of the CHANCE, EXP3 and EXP3.S strategies. It is also important to emphasise that for TO=0.05, the MD and the MAB strategies showed better results in terms of AR and SR when compared to the results obtained for the standard parameter set. Nonetheless, as stated previously, both EXP3 and EXP3.S showed poor performance in learning which portfolio allocation model had the highest average returns. Therefore, although the solution presented in Section 5.2.1 for $H = 20$ had a slightly higher turnover and lower Sharpe ratio than the solution presented here for $TO = 0{:}05$, both EXP3 and EXP3.S showed a better performance in learning which action was \optimal". In addition, these also showed superior performance in terms of maximum drawdown and risk diversi cation (ENC and ENCB). Since the portfolio allocation models in use, while being risk-based, do not prioritise the maximisation of the returns, the metrics that take risk and diversi cation into account are crucial for determining the goodness of a portfolio strategy. As such, the solution described in Section 5.2.1 for $H = 20$ is considered to be superior.

5.3 Market Generalisation

The results obtained previously only considered assets belonging to the American market. However, since portfolio strategies may present a di erent performance depending on the market where they are applied, the need to test the strategies analysed above in other markets becomes evident. Therefore, to have results with a higher degree of generalisation, three additional markets were selected: the Japanese, European and Hong Kong markets, which were chosen according to their importance in global terms, where the indices selected to represent these three markets are the *NIKKEI 225*, *EURO STOXX 50* and *HSI*, respectively. These are tested while using the turnover solution described in Section 5.2.1 for $H = 20$.

As for the American market, the assets that constitute the portfolio are the same ones that belong to the respective market index[9] between December 30, 2001, and December 31, 2021. However, since the Japanese market was not open between December 30, 2001, and January 3, 2002, data for these dates is not available. Therefore, for the Japanese market, the starting date considered is January 4, 2002.

In addition, the risk-free asset selected for the Japanese, European and Hong Kong markets are the 3-Month Yen LIBOR, 3-Month Euro LIBOR[10], and 3-Month Government Bond[11], respectively. The risk-free returns on these three markets also had the annualised returns for each daily value, which were transformed the same way as described for the American market in order to have daily return values. In Table 5.12, one may nd a descriptive summary of the markets under study in this section, and in Table 5.13, the parameter values for the CORE and CORE+ formulations for the same set of markets under the rst turnover solution with $H = 20$ and $TO = 0$.

Table 5.12: Descriptive summary for the Japanese, European and Hong Kong markets, which includes the representative market index, the number of assets N that constitute the portfolio, the risk-free asset considered for the particular market, the time period during which the historical returns were collected and the frequency in which these are.

Market	Index	N	Risk-Free Asset	Time Period	Frequency
Japanese	NIKKEI 225	197	3-M Yen LIBOR	2002-01-04 ! 2021-12-31	Daily
European	EURO STOXX 50	43	3-M Euro LIBOR	2001-12-31 ! 2021-12-31	Daily
Hong Kong	HSI	50	3-M Government Bond	2001-12-31 ! 2021-12-31	Daily

Table 5.14 shows the risk and return metrics values for the Japanese, European and Hong Kong markets, while $TC = 0$. The following remarks can be made:

- In all three markets, MD presented the highest value of average returns, Sharpe ratio, while MV obtained the lowest value of volatility and the highest maximum drawdown, except for the Japanese

[9] Despite being within the selected period, Tokio Marine Holdings, Inc. (8766.T), which belongs to the NIKKEI 225 index, was not taken into account since its share price showed irregular behaviour from September 28 to 29, 2005, during which the asset's value rose 520%, with the explanation for this event being unknown.

[10] The 3-Month Yen LIBOR's and 3-Month Euro LIBOR's daily annualised returns were taken from https://fred.stlouisfed.org/.

[11] The 3-Month Government Bond's daily annualised returns were taken from https://www.investing.com/.

Table 5.13: Parameter values for the CORE and CORE+ formulations on the Japanese, European and Hong Kong markets under the first turnover solution with H = 20 and TO = 0.

Market	L	H	TO	K	(EXP3)	(EXP3:S)	
Japanese	500	20	0	3	0.0932	0.2970	0.0045
European	500	20	0	3	0.0909	0.2909	0.0043
Hong Kong	500	20	0	3	0.0930	0.2964	0.0045

market, where EXP3 presented the highest maximum drawdown value instead;

- Between CHANCE and the two MAB strategies, the former outperformed the latter in terms of volatility in all three markets. As for the average returns and Sharpe ratio, CHANCE was superior in the Japanese market, while EXP3 was superior in the European and Hong Kong ones. At last, in terms of maximum drawdown, neither portfolio strategy was able to present itself as superior;

- In addition, for the Japanese and Hong Kong markets, not a single portfolio strategy have a p-value that rejects the null hypothesis that their Sharpe ratio is equal to EW's at a significance level lower or equal to 10%. In contrast, for the European markets, MD, CHANCE, EXP3 and EXP3.S reject the null hypothesis at the 5% significance level and ERC at the 10% significance level.

Table 5.14: Results of the risk and return metrics obtained by each portfolio strategy in the Japanese, European and Hong Kong markets, with H = 20, TO = 0 and TC = 0.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
Japanese Market							
AR (%)	0.0344	0.0205	0.0326	**0.0399**	0.0322	0.0316	0.0304
VO (%)	1.4503	**1.0396**	1.3458	1.2082	1.1421	1.1539	1.1524
SR	0.0237	0.0197	0.0243	**0.0330**	0.0282	0.0274	0.0264
	(1.00)	(0.67)	(0.65)	(0.26)	(0.42)	(0.49)	(0.62)
MDD (%)	-61.1323	-50.3532	-58.2754	-49.1087	-49.4183	**-48.3495**	-49.4612
European Market							
AR (%)	0.0391	0.0429	0.0404	**0.0577**	0.0462	0.0477	0.0473
VO (%)	1.3194	**1.0109**	1.2177	1.1633	1.0839	1.0870	1.0848
SR	0.0296	0.0424	0.0331	**0.0496**	0.0426	0.0439	0.0436
	(1.00)	(0.16)	(0.09)	(0.03)	(0.05)	(0.03)	(0.03)
MDD (%)	-60.3139	**-45.8917**	-56.5450	-49.1185	-50.6456	-50.1419	-50.1225
Hong Kong Market							
AR (%)	0.0355	0.0169	0.0302	**0.0398**	0.0298	0.0311	0.0310
VO (%)	1.3228	**0.9078**	1.1554	1.0543	0.9908	1.0097	1.0147
SR	0.0268	0.0186	0.0262	**0.0378**	0.0301	0.0308	0.0305
	(1.00)	(0.46)	(0.76)	(0.17)	(0.60)	(0.50)	(0.52)
MDD (%)	-62.2070	**-41.8758**	-57.0691	-50.3234	-48.7349	-49.7918	-50.4154

Table 5.15 shows the values of the risk diversification metrics for the Japanese, European and Hong Kong markets, while TC = 0. The following remarks can be made:

- Overall, EW had the best turnover and ENC values in all three markets, while ERC had the best performance in terms of ENCB;

- As for CHANCE and the two MAB strategies, when it comes to ENC and ENCB, each one was superior for a given market, i.e., EXP3 presented the highest value of the three strategies in both metrics for the Japanese, CHANCE for the European one and EXP3.S for the Hong Kong market. On the other hand, for the turnover EXP3.S was superior in all three markets.

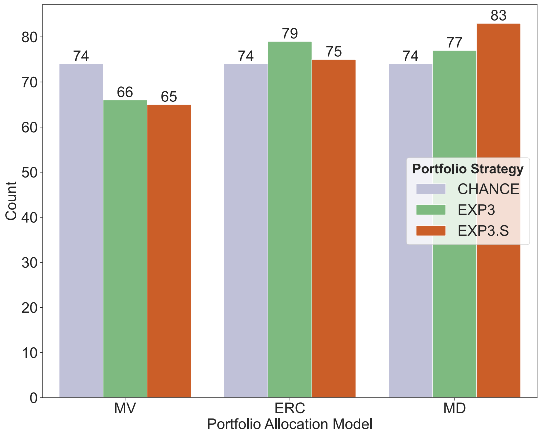
Table 5.15: Results of the risk diversification metrics obtained by each portfolio strategy in the Japanese, European and Hong Kong markets, with $H = 20$, $TO = 0$ and $TC = 0$.

	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
Japanese Market							
TRN (%)	**0.2705**	0.9580	0.2881	0.9593	5.4675	5.1772	4.7465
ENC	**196.7486**	13.0267	185.6654	20.0706	72.9131	77.1863	73.6118
ENCB	188.6581	13.0454	**196.6923**	19.8664	76.4684	81.0203	77.2382
European Market							
TRN (%)	**0.2146**	0.7039	0.2372	0.6570	3.9730	3.7319	3.3187
ENC	**42.9606**	8.6301	40.6408	12.0835	20.3672	20.3519	19.5908
ENCB	40.7993	8.6076	**42.8840**	12.0025	21.0749	21.0508	20.2440
Hong Kong Market							
TRN (%)	**0.2294**	0.5701	0.2466	0.5145	3.6056	3.3752	3.0169
ENC	**23.9753**	5.5129	22.0630	9.2693	12.2814	12.9772	13.2324
ENCB	22.2498	5.5261	**23.9617**	9.9424	13.1329	13.9250	14.2139

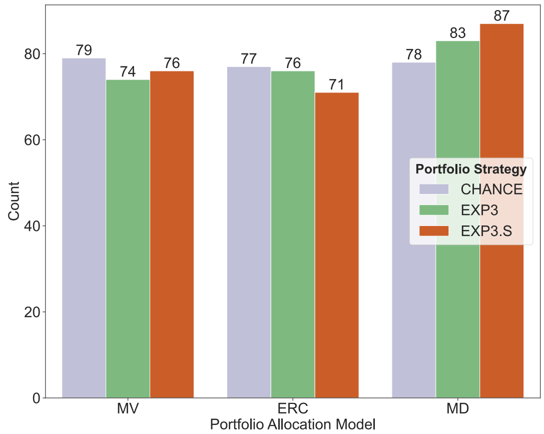
Thus, for $TC = 0$, it is noticeable that compared to CHANCE and EXP3.S, EXP3 had a better overall performance in the risk and return metrics as well as in risk diversification. Its superiority results from the fact that for the first set of metrics, this one was superior in the European and Hong Kong markets, and for the Japanese market, its Sharpe ratio, despite being lower than CHANCE's, the difference in value is not too significant. Furthermore, this one had the best maximum drawdown for the Japanese market and the second best for the European and Hong Kong markets. In addition, when it comes to risk diversification, it showed better overall performance among the three strategies since it was never the worst-performing strategy. In fact, it had the best risk diversification in the Japanese market, according to the ENC and ENCB results.

Moreover, both EXP3 and EXP3.S showed better Sharpe ratio results for the European and Hong Kong markets compared to the results that CHANCE was able to yield. However, according to Figure 5.6(a) (which represents the number of times CHANCE, EXP3 and EXP3.S selected each of the possible actions (MV, ERC and MD) for the Japanese market), the average return that the latter two yielded was inferior to the one obtained by CHANCE, which resulted in lower Sharpe ratios. This might be because despite having selected the “optimal” portfolio allocation model more times, at a particular time step, the one considered “optimal” might not have been the action that would have led to the highest reward. However, it was still selected since the probability of yielding the highest returns might have been the highest of the three available actions.

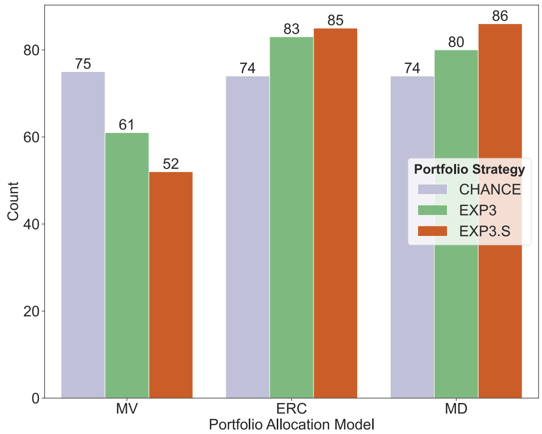
As previously stated, EXP3.S presented the lowest values in terms of turnover, while CHANCE had the highest ones. As a result, despite having lower average returns than both CHANCE and EXP3 for



(a) Japanese Market



(b) European Market

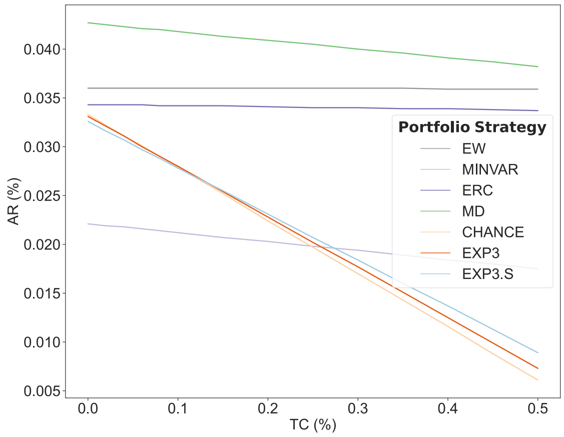


(c) Hong Kong Market

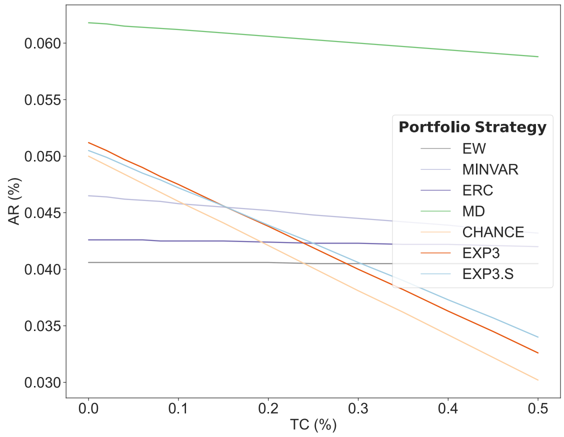
Figure 5.6: Number of times CHANCE, EXP3 and EXP3.S selected each of the available actions (MV, ERC and MD) on the Japanese, European and Hong Kong markets, for $H = 20$, $TO = 0$ and $TC = 0$.

$TC = 0$ in the Japanese market, according to Figure 5.7(a), as TC increases in value, this strategy is able to retain a more signi cant portion of its returns by eventually surpassing the average returns yielded by CHANCE and EXP3. The same happens for the European and Hong Kong markets; however, at $TC = 0$, its average return was only inferior to EXP3's, Figures 5.7(b) and 5.7(c). Additionally, it is important to point out that, despite EXP3 presenting the best overall risk diversi cation, the EXP3.S results are not signi cantly inferior, since for the Hong Kong market, this one yielded the best results in ENC and ENCB, and in the Japanese, it was the second best.

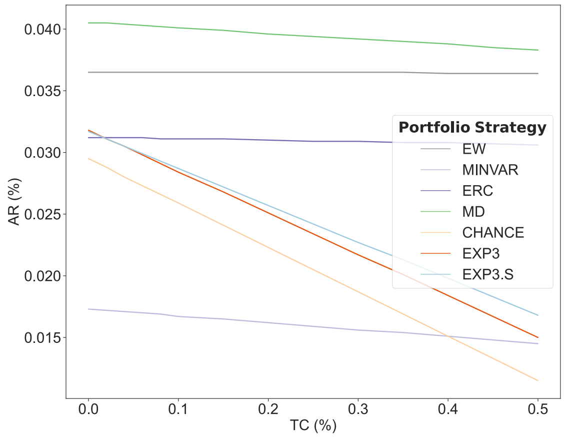
Furthermore, analysing the performance of CHANCE and the two MAB strategies when the proportional transaction costs are di erent from zero against EW, MV, ERC and MD, Figure 5.7, it is noticeable at rst that, in all three markets, MD has the better overall performance in retaining its



(a) Japanese Market



(b) European Market



(c) Hong Kong Market

Figure 5.7: Average returns for the EW, MV, ERC, MD, CHANCE, EXP3 and EXP3.S strategies on the Japanese, European and Hong Kong markets for $TC \in [0, 0.005]$, $H = 20$ and $TO = 0$.

returns as TC increases in value. At the same time, the distance between EXP3.S's average returns and those of CHANCE and EXP3 tends to increase linearly with TC.

Additionally, for the Japanese market, Figure 5.7(a), CHANCE and the two MAB strategies generate higher average returns than MV till $TC \le 0.0030$. However, after this value, their average returns continues decreasing faster than MV, given the higher turnover values these possess. For the European market, Figure 5.7(b), up until $TC \le 0.0010$, CHANCE and the two MAB strategies show higher average returns than EW, MV and ERC while only falling behind MD. Then, however, since these have a superior turnover, between TC of 0.0010 and 0.0030, their average returns start to trend lower than EW, MV and ERC. Last, for the Hong Kong market, Figure 5.7(c), for the TC values tested, both EXP3 and EXP3.S always yield average returns superior to MV, while CHANCE can only do it till a TC value of 0.0040, as after that, its average returns are lower than MV. Additionally, for TC values lower than 0.0002, both EXP3 and EXP3.S yield higher average returns than ERC.

Finally, according to the results obtained above, it is possible to conclude that when $TC = 0$, the EXP3 strategy presents better results than both CHANCE and EXP3.S. However, if the TC values are different from zero, EXP3.S presents itself as a better solution, particularly for higher TC values, since it offers a lower turnover. Additionally, the conclusion obtained for the American market as the EXP3.S being the best overall strategy for $TC \geq 0$ cannot be generalised for other markets if $TC = 0$; Nevertheless, for $TC > 0$, that conclusion is still verified for the three new markets tested. However, both EXP3 and EXP3.S strategies are still very limited on the window of TC values where these are worth being used as a portfolio strategy.

Chapter 6

Conclusion

6.1 General Overview

There are many portfolio allocation models available in the literature, with those focusing on the minimisation of risk being no exception (even though their exploration only started in a recent past). Each of these models has qualities that others do not. For example, of those explored in this study, MV has the lowest volatility in most of the tests conducted, while MD has the highest average returns, and ERC has the lowest turnover and ENCB. In addition, given the model's metrics results, their performance tends to di er according to the market under consideration. As a result, given that the portfolio's weights are optimised periodically, this thesis studies the applicability of algorithms typically used to solve MAB problems in selecting the \optimal" portfolio allocation model in each of the various allocation periods. From its application, it is expected that the result obtained in each of the metrics will be equal or superior to the average of the various portfolio allocation models that the MAB algorithm has at its disposal to select from in each allocation period.

Moreover, the EXP3 and EXP3.S algorithms were chosen to select a portfolio allocation model at each allocation period. These were chosen given that they were initially developed to solve AB problems, a generalisation of the MAB problems in which no statistical assumptions of the generation of the rewards are made. These two algorithms use a weight vector where each entry has a value for the weight of each portfolio allocation model. After a model has been selected at each allocation period, the weight value for this model is updated with the generated reward, which will directly impact the probability of selecting this model for the next allocation period. Thus, since the reward function considers the portfolio return that the model generated between two allocation periods, for which the distribution is unknown, it becomes necessary not to make any statistical assumptions regarding the rewards distribution. Therefore, both EXP3 and EXP3.S were suited to eld this requirement.

Thus, in order to verify if EXP3 and EXP3.S are indeed viable solutions to the described problem, these are applied to real-world data, from which it is possible to evaluate their performance according to a set of metrics, where these evaluate the portfolio strategy's return, risk, and risk diversi cation. In addition, their results are compared with those obtained from the portfolio allocation models they have at their disposal (MV, ERC and MD), with EW (benchmark model), and with CHANCE (an alternative to EXP3 and EXP3.S that selects at each allocation period a portfolio allocation model according to a uniform probability distribution). Regarding the parameters used by EXP3 and EXP3.S, these are always deterministic.

Considering a portfolio where its constituents are traded in the American market, the rst analysis of results is done considering a set of standard parameters, i.e., daily allocation, no turnover restriction and no proportional transaction costs. From the results obtained on the various metrics, the problem initially disclosed was solved as both EXP3 and EXP3.S presented results superior/equal to the average results obtained by MV, ERC and MD on the selected metrics (except for turnover), with EXP3.S showing better results in terms of Sharpe ratio and risk diversification than EXP3. Furthermore, both strategies demonstrated superiority over the CHANCE portfolio strategy. Despite these results, the turnover that resulted from CHANCE, EXP3, and EXP3.S was significantly high, making it infeasible to use these three portfolio strategies in a scenario where TC assumes values larger than 0.

Thus, in order to decrease the turnover generated by EXP3 and EXP3.S, two solutions were tested. The first consists of increasing the interval between two allocation periods, while the second introduces a restriction on the maximum turnover value each asset can present between two allocation periods. Both solutions effectively reduced the turnover value, making the two portfolio strategies usable when the proportional transaction costs were considered (i.e., different than zero). On the other hand, if the proportional transaction costs were not accounted for, both EXP3 and EXP3.S had better overall performance using the standard parameters. Additionally, despite having a slightly lower turnover and maximum drawdown, the solution presented in Section 5.2.1 with $H = 20$ proved to be superior to the solution presented in Section 5.2.1 with $TO = 0.05$ (values for which each solution obtained the best overall performance) since both EXP3 and EXP3.S presented a better ability to detect the "optimal" portfolio allocation model. Besides that, they could also achieve superior performance in risk diversification, particularly in ENC and ENCB.

Furthermore, in order to verify if the results obtained so far were generalisable to markets other than the American one, the Japanese, European and Hong Kong markets were tested out, for which the best turnover solution obtained in Section 5.2 was applied. According to the results, it is possible to verify that the solution selected as the better one for the American market is also able to achieve its intent of having a turnover low enough on EXP3 and EXP3.S for these to be used in practical terms on these three new markets. However, it was verified that when the proportional transaction costs were not considered, the superiority EXP3.S have shown in the American market was not generalisable to other markets, as EXP3 showed better performance in these three new markets (nevertheless, the results obtained via EXP3.S were not too far apart from those of EXP3). On the other hand, when the proportional transaction costs were taken into account, given that EXP3.S had a lower turnover than EXP3, the former had a superior performance for this setting.

Altogether, it can be concluded that if proportional transaction costs are non-existent, then the best overall results are obtained using the standard parameters. On the other hand, if proportional transaction costs are taken into account, then a solution to decrease the turnover is needed. Therefore, it was concluded that the one presented in Section 5.2.1 with $H = 20$ led to the most satisfactory performance results. However, despite the satisfactory performance, these strategies are not feasible for high proportional transaction costs. Furthermore, if the investor aims to maximise the Sharpe ratio, these strategies are not indicated because the MD portfolio allocation model presented the best results in this metric for the four markets analysed. On the other hand, if the investor aims to achieve a Sharpe ratio with a significant value and at the same time a good risk diversification (ENC and ENCB), then EXP3 and EXP3.S reveal themselves as sound compromise solutions for relatively low proportional transaction costs.

6.2 Future Work

In order to fill some gaps that might not be yet clear, future avenues of work that build upon this dissertation are here proposed.

The rewards obtained from the considered reward function are generated based on the return obtained between the allocation period in which the portfolio allocation model was applied and the following (in which the return is obtained). Thus, it is suggested to investigate if the performance of the MAB algorithms would change if the proportional transaction costs were taken into account on the return used to compute the reward at each allocation period. It is also suggested to use a different metric to compute the rewards, such as the Sharpe ratio, as it presents a higher level of information since it considers both average returns and volatility. Additionally, it is suggested to use the rolling window method to compute the rewards, for which a window length would be selected. This addition may remove existing noise in old data and lead to faster learning by EXP3 and EXP3.S of the "optimal" portfolio allocation model for a given time horizon.

Moreover, the value of the parameter used in EXP3 and EXP3.S, and for the latter one, were assigned deterministically. Then, one suggests testing other values for these two parameters and verifying if it positively impacts the learning rate of EXP3 and EXP3.S and their overall performance as portfolio strategies.

Finally, in this dissertation, both EXP3 and EXP3.S had only three different portfolio allocation models from which they could choose. Therefore, it is suggested to add new models with different characteristics to this set and verify if the performance obtained by EXP3 and EXP3.S is limited by the three portfolio allocation models previously used or if they benefit from adding new models. For example, approaches such as risk budgeting [8] or diversified risk parity [36] are widely used in practice, as they provide elegant and systematic methodologies to approach diversified portfolio construction.

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Appendix A

Complementary Results

Table A.1: Average returns of each portfolio strategy in the American market for di erent TC values, H = 5 and TO = 0.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0547	0.0342	0.0499	0.0628	0.0500	0.0512	0.0519
0.02	0.0546	0.0337	0.0498	0.0624	0.0456	0.0469	0.0482
0.04	0.0545	0.0332	0.0497	0.0620	0.0412	0.0426	0.0445
0.06	0.0544	0.0326	0.0496	0.0615	0.0367	0.0383	0.0408
0.08	0.0543	0.0321	0.0495	0.0611	0.0323	0.0340	0.0371
0.10	0.0542	0.0315	0.0494	0.0607	0.0279	0.0297	0.0335
0.15	0.0539	0.0301	0.0491	0.0597	0.0168	0.0190	0.0243
0.20	0.0537	0.0288	0.0489	0.0587	0.0058	0.0083	0.0151
0.25	0.0534	0.0274	0.0486	0.0576	-0.0053	-0.0024	0.0059
0.30	0.0532	0.0260	0.0483	0.0566	-0.0163	-0.0131	-0.0033
0.35	0.0529	0.0247	0.0481	0.0556	-0.0274	-0.0239	-0.0125
0.40	0.0527	0.0233	0.0478	0.0546	-0.0384	-0.0346	-0.0217
0.45	0.0524	0.0219	0.0475	0.0535	-0.0495	-0.0453	-0.0309
0.50	0.0522	0.0206	0.0473	0.0525	-0.0606	-0.0560	-0.0401

Table A.2: Average returns of each portfolio strategy in the American market for di erent TC values, H = 20 and TO = 0.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0536	0.0332	0.0491	0.0609	0.0477	0.0492	0.0492
0.02	0.0535	0.0329	0.0490	0.0607	0.0465	0.0481	0.0482
0.04	0.0534	0.0326	0.0489	0.0605	0.0454	0.0470	0.0473
0.06	0.0534	0.0324	0.0489	0.0602	0.0442	0.0459	0.0463
0.08	0.0533	0.0321	0.0488	0.0600	0.0431	0.0449	0.0453
0.10	0.0533	0.0318	0.0488	0.0597	0.0419	0.0438	0.0444
0.15	0.0532	0.0312	0.0486	0.0592	0.0390	0.0411	0.0419
0.20	0.0530	0.0305	0.0485	0.0586	0.0362	0.0384	0.0395
0.25	0.0529	0.0298	0.0483	0.0580	0.0333	0.0356	0.0371
0.30	0.0528	0.0292	0.0482	0.0574	0.0304	0.0329	0.0346
0.35	0.0526	0.0285	0.0481	0.0568	0.0275	0.0302	0.0322
0.40	0.0525	0.0278	0.0479	0.0562	0.0247	0.0275	0.0298
0.45	0.0524	0.0272	0.0478	0.0556	0.0218	0.0248	0.0274
0.50	0.0522	0.0265	0.0476	0.0550	0.0189	0.0221	0.0249

Table A.3: Average returns of each portfolio strategy in the American market for different TC values, $H = 1$ and $TO = 0.20$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0555	0.0317	0.0505	0.0613	0.0410	0.0428	0.0460
0.02	0.0552	0.0314	0.0503	0.0609	0.0392	0.0409	0.0443
0.04	0.0550	0.0310	0.0501	0.0605	0.0373	0.0390	0.0425
0.06	0.0548	0.0307	0.0499	0.0601	0.0355	0.0371	0.0408
0.08	0.0546	0.0304	0.0497	0.0598	0.0336	0.0352	0.0390
0.10	0.0544	0.0300	0.0494	0.0594	0.0318	0.0333	0.0373
0.15	0.0539	0.0292	0.0489	0.0585	0.0271	0.0286	0.0329
0.20	0.0533	0.0283	0.0484	0.0575	0.0225	0.0239	0.0285
0.25	0.0528	0.0275	0.0478	0.0566	0.0179	0.0191	0.0241
0.30	0.0522	0.0266	0.0473	0.0556	0.0132	0.0144	0.0197
0.35	0.0517	0.0257	0.0467	0.0547	0.0086	0.0097	0.0154
0.40	0.0512	0.0249	0.0462	0.0537	0.0040	0.0050	0.0110
0.45	0.0506	0.0240	0.0457	0.0528	-0.0006	0.0002	0.0066
0.50	0.0501	0.0232	0.0451	0.0519	-0.0053	-0.0045	0.0022

Table A.4: Average returns of each portfolio strategy in the American market for different TC values, $H = 1$ and $TO = 0.15$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0555	0.0326	0.0505	0.0671	0.0445	0.0458	0.0483
0.02	0.0552	0.0323	0.0503	0.0668	0.0431	0.0444	0.0470
0.04	0.0550	0.0320	0.0501	0.0665	0.0416	0.0429	0.0457
0.06	0.0548	0.0317	0.0499	0.0662	0.0402	0.0414	0.0444
0.08	0.0546	0.0314	0.0497	0.0659	0.0387	0.0399	0.0431
0.10	0.0544	0.0311	0.0494	0.0656	0.0373	0.0384	0.0418
0.15	0.0539	0.0304	0.0489	0.0648	0.0336	0.0347	0.0385
0.20	0.0533	0.0296	0.0484	0.0640	0.0300	0.0310	0.0353
0.25	0.0528	0.0288	0.0478	0.0633	0.0263	0.0272	0.0321
0.30	0.0522	0.0281	0.0473	0.0625	0.0227	0.0235	0.0288
0.35	0.0517	0.0273	0.0468	0.0617	0.0191	0.0198	0.0256
0.40	0.0512	0.0265	0.0462	0.0609	0.0154	0.0161	0.0223
0.45	0.0506	0.0258	0.0457	0.0602	0.0118	0.0123	0.0191
0.50	0.0501	0.0250	0.0451	0.0594	0.0081	0.0086	0.0159

Table A.5: Average returns of each portfolio strategy in the American market for di erent TC values, $H = 1$ and $TO = 0{:}10$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0555	0.0332	0.0505	0.0777	0.0510	0.0519	0.0533
0.02	0.0552	0.0330	0.0502	0.0775	0.0499	0.0508	0.0524
0.04	0.0550	0.0327	0.0500	0.0773	0.0489	0.0498	0.0515
0.06	0.0548	0.0325	0.0498	0.0771	0.0478	0.0487	0.0506
0.08	0.0546	0.0322	0.0496	0.0769	0.0468	0.0476	0.0497
0.10	0.0544	0.0320	0.0494	0.0767	0.0458	0.0465	0.0488
0.15	0.0539	0.0314	0.0488	0.0761	0.0432	0.0439	0.0465
0.20	0.0533	0.0308	0.0483	0.0756	0.0405	0.0412	0.0442
0.25	0.0528	0.0302	0.0478	0.0750	0.0379	0.0385	0.0420
0.30	0.0522	0.0296	0.0472	0.0745	0.0353	0.0358	0.0397
0.35	0.0517	0.0290	0.0467	0.0739	0.0327	0.0331	0.0374
0.40	0.0512	0.0285	0.0462	0.0734	0.0301	0.0305	0.0351
0.45	0.0506	0.0279	0.0456	0.0729	0.0275	0.0278	0.0329
0.50	0.0501	0.0273	0.0451	0.0723	0.0249	0.0251	0.0306

Table A.6: Average returns of each portfolio strategy in the American market for di erent TC values, $H = 1$ and $TO = 0{:}05$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0555	0.0359	0.0503	0.0961	0.0655	0.0668	0.0674
0.02	0.0552	0.0357	0.0501	0.0960	0.0649	0.0662	0.0669
0.04	0.0550	0.0356	0.0499	0.0959	0.0644	0.0656	0.0664
0.06	0.0548	0.0354	0.0497	0.0957	0.0638	0.0650	0.0660
0.08	0.0546	0.0353	0.0495	0.0956	0.0632	0.0644	0.0655
0.10	0.0544	0.0352	0.0492	0.0955	0.0627	0.0638	0.0650
0.15	0.0539	0.0348	0.0487	0.0952	0.0612	0.0623	0.0638
0.20	0.0533	0.0345	0.0482	0.0949	0.0598	0.0609	0.0626
0.25	0.0528	0.0341	0.0477	0.0946	0.0584	0.0594	0.0614
0.30	0.0522	0.0338	0.0472	0.0944	0.0570	0.0579	0.0602
0.35	0.0517	0.0334	0.0466	0.0941	0.0555	0.0564	0.0590
0.40	0.0512	0.0331	0.0461	0.0938	0.0541	0.0549	0.0578
0.45	0.0506	0.0327	0.0456	0.0935	0.0527	0.0535	0.0567
0.50	0.0501	0.0324	0.0451	0.0932	0.0513	0.0520	0.0555

Table A.7: Average returns of each portfolio strategy in the Japanese market for different TC values, $H = 20$ and $TO = 0$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0344	0.0205	0.0326	0.0399	0.0322	0.0316	0.0304
0.02	0.0344	0.0203	0.0326	0.0397	0.0311	0.0305	0.0295
0.04	0.0343	0.0201	0.0325	0.0395	0.0300	0.0295	0.0285
0.06	0.0343	0.0199	0.0325	0.0393	0.0289	0.0284	0.0276
0.08	0.0342	0.0197	0.0324	0.0391	0.0278	0.0274	0.0266
0.10	0.0341	0.0195	0.0323	0.0389	0.0267	0.0264	0.0257
0.15	0.0340	0.0190	0.0322	0.0384	0.0240	0.0238	0.0233
0.20	0.0339	0.0185	0.0320	0.0379	0.0212	0.0212	0.0209
0.25	0.0337	0.0180	0.0319	0.0374	0.0185	0.0186	0.0185
0.30	0.0336	0.0175	0.0317	0.0370	0.0157	0.0160	0.0161
0.35	0.0334	0.0170	0.0316	0.0365	0.0130	0.0134	0.0137
0.40	0.0333	0.0165	0.0314	0.0360	0.0102	0.0107	0.0113
0.45	0.0331	0.0161	0.0312	0.0355	0.0075	0.0081	0.0090
0.50	0.0330	0.0156	0.0311	0.0350	0.0047	0.0055	0.0066

Table A.8: Average returns of each portfolio strategy in the European market for different TC values, $H = 20$ and $TO = 0$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0391	0.0429	0.0404	0.0577	0.0462	0.0477	0.0473
0.02	0.0391	0.0428	0.0403	0.0576	0.0454	0.0470	0.0466
0.04	0.0390	0.0426	0.0403	0.0575	0.0446	0.0462	0.0459
0.06	0.0390	0.0425	0.0402	0.0573	0.0438	0.0455	0.0453
0.08	0.0389	0.0423	0.0402	0.0572	0.0430	0.0447	0.0446
0.10	0.0389	0.0422	0.0401	0.0571	0.0422	0.0440	0.0439
0.15	0.0388	0.0418	0.0400	0.0567	0.0402	0.0421	0.0423
0.20	0.0386	0.0415	0.0398	0.0564	0.0382	0.0402	0.0406
0.25	0.0385	0.0411	0.0397	0.0560	0.0362	0.0383	0.0389
0.30	0.0384	0.0407	0.0396	0.0557	0.0342	0.0365	0.0372
0.35	0.0383	0.0404	0.0395	0.0554	0.0322	0.0346	0.0356
0.40	0.0382	0.0400	0.0393	0.0550	0.0302	0.0327	0.0339
0.45	0.0380	0.0396	0.0392	0.0547	0.0282	0.0308	0.0322
0.50	0.0379	0.0393	0.0391	0.0543	0.0262	0.0290	0.0306

Table A.9: Average returns of each portfolio strategy in the Hong Kong market for di erent TC values, $H = 20$ and $TO = 0$.

TC (%)	Average Returns (%)						
	EW	MV	ERC	MD	CHANCE	EXP3	EXP3.S
0.00	0.0355	0.0169	0.0302	0.0398	0.0298	0.0311	0.0310
0.02	0.0354	0.0168	0.0302	0.0397	0.0291	0.0304	0.0304
0.04	0.0354	0.0167	0.0301	0.0396	0.0284	0.0297	0.0297
0.06	0.0353	0.0166	0.0301	0.0395	0.0277	0.0290	0.0291
0.08	0.0353	0.0164	0.0300	0.0394	0.0269	0.0284	0.0285
0.10	0.0352	0.0163	0.0300	0.0393	0.0262	0.0277	0.0279
0.15	0.0351	0.0160	0.0298	0.0390	0.0244	0.0260	0.0264
0.20	0.0350	0.0157	0.0297	0.0387	0.0226	0.0243	0.0249
0.25	0.0349	0.0154	0.0296	0.0385	0.0208	0.0226	0.0234
0.30	0.0347	0.0151	0.0294	0.0382	0.0189	0.0209	0.0218
0.35	0.0346	0.0148	0.0293	0.0379	0.0171	0.0192	0.0203
0.40	0.0345	0.0145	0.0292	0.0377	0.0153	0.0175	0.0188
0.45	0.0344	0.0143	0.0290	0.0374	0.0135	0.0158	0.0173
0.50	0.0342	0.0140	0.0289	0.0371	0.0117	0.0141	0.0158

Appendix B

Portfolio Constituents Description

Tables B.1, B.2, B.3 and B.4 contain additional information regarding the assets that belong to each of the portfolios tested in the American, Japanese, European and Hong Kong, respectively. In particular, for each portfolio's constituent, the table contains the company's name, the sector in which it operates, and the ticker symbol of the asset in the market where it is listed[1].

Table B.1: Description of the constituents that belong to the portfolio traded in the American market. For each constituent, the respective company name, sector and symbol are given.

Company Name	Sector	Symbol
Air Products and Chemicals, Inc.	Basic Materials	APD
Albemarle Corporation	Basic Materials	ALB
DuPont de Nemours, Inc.	Basic Materials	DD
Eastman Chemical Company	Basic Materials	EMN
Ecolab Inc.	Basic Materials	ECL
FMC Corporation	Basic Materials	FMC
Freeport-McMoRan Inc.	Basic Materials	FCX
International Flavors & Fragrances Inc.	Basic Materials	IFF
Linde plc	Basic Materials	LIN
Martin Marietta Materials, Inc.	Basic Materials	MLM
The Mosaic Company	Basic Materials	MOS
Newmont Corporation	Basic Materials	NEM
Nucor Corporation	Basic Materials	NUE
PPG Industries, Inc.	Basic Materials	PPG
The Sherwin-Williams Company	Basic Materials	SHW
Vulcan Materials Company	Basic Materials	VMC
Activision Blizzard, Inc.	Communication Services	ATVI
AT&T Inc.	Communication Services	T
Comcast Corporation	Communication Services	CMCSA
DISH Network Corporation	Communication Services	DISH
Electronic Arts Inc.	Communication Services	EA
The Interpublic Group of Companies, Inc.	Communication Services	IPG
Lumen Technologies, Inc.	Communication Services	LUMN
Match Group, Inc.	Communication Services	MTCH
Omnicom Group Inc.	Communication Services	OMC
Take-Two Interactive Software, Inc.	Communication Services	TTWO
Verizon Communications Inc.	Communication Services	VZ
The Walt Disney Company	Communication Services	DIS
Advance Auto Parts, Inc.	Consumer Cyclical	AAP
Amazon.com, Inc.	Consumer Cyclical	AMZN

[1] The information was taken from https://finance.yahoo.com/.

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AutoZone, Inc.	Consumer Cyclical	AZO
Ball Corporation	Consumer Cyclical	BLL
Bath & Body Works, Inc.	Consumer Cyclical	BBWI
Best Buy Co., Inc.	Consumer Cyclical	BBY
Booking Holdings Inc.	Consumer Cyclical	BKNG
BorgWarner Inc.	Consumer Cyclical	BWA
CarMax, Inc.	Consumer Cyclical	KMX
Carnival Corporation & plc	Consumer Cyclical	CCL
Copart, Inc.	Consumer Cyclical	CPRT
D.R. Horton, Inc.	Consumer Cyclical	DHI
Darden Restaurants, Inc.	Consumer Cyclical	DRI
eBay Inc.	Consumer Cyclical	EBAY
Ford Motor Company	Consumer Cyclical	F
The Gap, Inc.	Consumer Cyclical	GPS
Genuine Parts Company	Consumer Cyclical	GPC
Hasbro, Inc.	Consumer Cyclical	HAS
The Home Depot, Inc.	Consumer Cyclical	HD
International Paper Company	Consumer Cyclical	IP
Lennar Corporation	Consumer Cyclical	LEN
Lowe's Companies, Inc.	Consumer Cyclical	LOW
Marriott International, Inc.	Consumer Cyclical	MAR
McDonald's Corporation	Consumer Cyclical	MCD
MGM Resorts International	Consumer Cyclical	MGM
Mohawk Industries, Inc.	Consumer Cyclical	MHK
NIKE, Inc.	Consumer Cyclical	NKE
NVR, Inc.	Consumer Cyclical	NVR
O'Reilly Automotive, Inc.	Consumer Cyclical	ORLY
Packaging Corporation of America	Consumer Cyclical	PKG
Penn National Gaming, Inc.	Consumer Cyclical	PENN
Pool Corporation	Consumer Cyclical	POOL
PulteGroup, Inc.	Consumer Cyclical	PHM
PVH Corp.	Consumer Cyclical	PVH
Ralph Lauren Corporation	Consumer Cyclical	RL
Rollins, Inc.	Consumer Cyclical	ROL
Ross Stores, Inc.	Consumer Cyclical	ROST
Royal Caribbean Cruises Ltd.	Consumer Cyclical	RCL
Sealed Air Corporation	Consumer Cyclical	SEE
Starbucks Corporation	Consumer Cyclical	SBUX
Tapestry, Inc.	Consumer Cyclical	TPR
The TJX Companies, Inc.	Consumer Cyclical	TJX
Tractor Supply Company	Consumer Cyclical	TSCO
V.F. Corporation	Consumer Cyclical	VFC
Whirlpool Corporation	Consumer Cyclical	WHR
Yum! Brands, Inc.	Consumer Cyclical	YUM

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Altria Group, Inc.	Consumer Defensive	MO
Archer-Daniels-Midland Company	Consumer Defensive	ADM
Brown-Forman Corporation	Consumer Defensive	BF-B
Campbell Soup Company	Consumer Defensive	CPB
Church & Dwight Co., Inc.	Consumer Defensive	CHD
The Clorox Company	Consumer Defensive	CLX
The Coca-Cola Company	Consumer Defensive	KO
Colgate-Palmolive Company	Consumer Defensive	CL
Conagra Brands	Consumer Defensive	CAG
Constellation Brands, Inc.	Consumer Defensive	STZ
Costco Wholesale Corporation	Consumer Defensive	COST
Dollar Tree, Inc.	Consumer Defensive	DLTR
The Estee Lauder Companies Inc.	Consumer Defensive	EL
General Mills, Inc.	Consumer Defensive	GIS
The Hershey Company	Consumer Defensive	HSY
Hormel Foods Corporation	Consumer Defensive	HRL
The J. M. Smucker Company	Consumer Defensive	SJM
Kellogg Company	Consumer Defensive	K
Kimberly-Clark Corporation	Consumer Defensive	KMB
The Kroger Co.	Consumer Defensive	KR
McCormick & Company, Incorporated	Consumer Defensive	MKC
Molson Coors Beverage Company	Consumer Defensive	TAP
Mondelez International, Inc.	Consumer Defensive	MDLZ
Monster Beverage Corporation	Consumer Defensive	MNST
Newell Brands Inc.	Consumer Defensive	NWL
PepsiCo, Inc.	Consumer Defensive	PEP
The Procter & Gamble Company	Consumer Defensive	PG
Sysco Corporation	Consumer Defensive	SYY
Target Corporation	Consumer Defensive	TGT
Tyson Foods, Inc.	Consumer Defensive	TSN
Walmart Inc.	Consumer Defensive	WMT
APA Corporation	Energy	APA
Baker Hughes Company	Energy	BKR
Chevron Corporation	Energy	CVX
ConocoPhillips	Energy	COP
Coterra Energy Inc.	Energy	CTRA
Devon Energy Corporation	Energy	DVN
EOG Resources, Inc.	Energy	EOG
Exxon Mobil Corporation	Energy	XOM
Halliburton Company	Energy	HAL
Hess Corporation	Energy	HES
Marathon Oil Corporation	Energy	MRO
Occidental Petroleum Corporation	Energy	OXY
ONEOK, Inc.	Energy	OKE

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Pioneer Natural Resources Company	Energy	PXD
Schlumberger Limited	Energy	SLB
Valero Energy Corporation	Energy	VLO
The Williams Companies, Inc.	Energy	WMB
A ac Incorporated	Financial Services	AFL
The Allstate Corporation	Financial Services	ALL
American Express Company	Financial Services	AXP
American International Group, Inc.	Financial Services	AIG
Aon plc	Financial Services	AON
Arthur J. Gallagher & Co.	Financial Services	AJG
Bank of America Corporation	Financial Services	BAC
Berkshire Hathaway Inc.	Financial Services	BRK-B
BlackRock, Inc.	Financial Services	BLK
The Bank of New York Mellon Corporation	Financial Services	BK
Brown & Brown, Inc.	Financial Services	BRO
Capital One Financial Corporation	Financial Services	COF
The Charles Schwab Corporation	Financial Services	SCHW
Chubb Limited	Financial Services	CB
Cincinnati Financial Corporation	Financial Services	CINF
Citigroup Inc.	Financial Services	C
Comerica Incorporated	Financial Services	CMA
Everest Re Group, Ltd.	Financial Services	RE
FactSet Research Systems Inc.	Financial Services	FDS
Fifth Third Bancorp	Financial Services	FITB
Franklin Resources, Inc.	Financial Services	BEN
Globe Life Inc.	Financial Services	GL
The Goldman Sachs Group, Inc.	Financial Services	GS
The Hartford Financial Services Group, Inc.	Financial Services	HIG
Huntington Bancshares Incorporated	Financial Services	HBAN
Invesco Ltd.	Financial Services	IVZ
JPMorgan Chase & Co.	Financial Services	JPM
KeyCorp	Financial Services	KEY
Lincoln National Corporation	Financial Services	LNC
Loews Corporation	Financial Services	L
M&T Bank Corporation	Financial Services	MTB
Marsh & McLennan Companies, Inc.	Financial Services	MMC
MetLife, Inc.	Financial Services	MET
Moody's Corporation	Financial Services	MCO
Morgan Stanley	Financial Services	MS
Northern Trust Corporation	Financial Services	NTRS
People's United Financial, Inc.	Financial Services	PBCT
The PNC Financial Services Group, Inc.	Financial Services	PNC
Principal Financial Group, Inc.	Financial Services	PFG
The Progressive Corporation	Financial Services	PGR

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Prudential Financial, Inc.	Financial Services	PRU
Raymond James Financial, Inc.	Financial Services	RJF
Regions Financial Corporation	Financial Services	RF
S&P Global Inc.	Financial Services	SPGI
State Street Corporation	Financial Services	STT
SVB Financial Group	Financial Services	SIVB
T. Rowe Price Group, Inc.	Financial Services	TROW
The Travelers Companies, Inc.	Financial Services	TRV
Truist Financial Corporation	Financial Services	TFC
U.S. Bancorp	Financial Services	USB
W. R. Berkley Corporation	Financial Services	WRB
Wells Fargo & Company	Financial Services	WFC
Willis Towers Watson	Financial Services	WLTW
Zions Bancorporation, National Association	Financial Services	ZION
Abbott Laboratories	Healthcare	ABT
Abiomed, Inc.	Healthcare	ABMD
Agilent Technologies, Inc.	Healthcare	A
Align Technology, Inc.	Healthcare	ALGN
AmerisourceBergen Corporation	Healthcare	ABC
Amgen Inc.	Healthcare	AMGN
Anthem, Inc.	Healthcare	ANTM
Baxter International Inc.	Healthcare	BAX
Becton, Dickinson and Company	Healthcare	BDX
Bio-Rad Laboratories, Inc.	Healthcare	BIO
Bio-Techne Corporation	Healthcare	TECH
Biogen Inc.	Healthcare	BIIB
Boston Scienti c Corporation	Healthcare	BSX
Bristol-Myers Squibb Company	Healthcare	BMY
Cardinal Health, Inc.	Healthcare	CAH
Centene Corporation	Healthcare	CNC
Cerner Corporation	Healthcare	CERN
Charles River Laboratories International, Inc.	Healthcare	CRL
Cigna Corporation	Healthcare	CI
The Cooper Companies, Inc.	Healthcare	COO
CVS Health Corporation	Healthcare	CVS
Danaher Corporation	Healthcare	DHR
DaVita Inc.	Healthcare	DVA
DENTSPLY SIRONA Inc.	Healthcare	XRAY
Edwards Lifesciences Corporation	Healthcare	EW
Gilead Sciences, Inc.	Healthcare	GILD
Henry Schein, Inc.	Healthcare	HSIC
Hologic, Inc.	Healthcare	HOLX
Humana Inc.	Healthcare	HUM
IDEXX Laboratories, Inc.	Healthcare	IDXX

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Illumina, Inc.	Healthcare	ILMN
Incyte Corporation	Healthcare	INCY
Intuitive Surgical, Inc.	Healthcare	ISRG
Johnson & Johnson	Healthcare	JNJ
Laboratory Corporation of America Holdings	Healthcare	LH
Eli Lilly and Company	Healthcare	LLY
McKesson Corporation	Healthcare	MCK
Medtronic plc	Healthcare	MDT
Merck & Co., Inc.	Healthcare	MRK
Mettler-Toledo International Inc.	Healthcare	MTD
PerkinElmer, Inc.	Healthcare	PKI
P zer Inc.	Healthcare	PFE
Quest Diagnostics Incorporated	Healthcare	DGX
Regeneron Pharmaceuticals, Inc.	Healthcare	REGN
ResMed Inc.	Healthcare	RMD
STERIS plc	Healthcare	STE
Stryker Corporation	Healthcare	SYK
Tele ex Incorporated	Healthcare	TFX
Thermo Fisher Scienti c Inc.	Healthcare	TMO
UnitedHealth Group Incorporated	Healthcare	UNH
Universal Health Services, Inc.	Healthcare	UHS
Vertex Pharmaceuticals Incorporated	Healthcare	VRTX
Viatris Inc.	Healthcare	VTRS
Walgreens Boots Alliance, Inc.	Healthcare	WBA
Waters Corporation	Healthcare	WAT
West Pharmaceutical Services, Inc.	Healthcare	WST
Zimmer Biomet Holdings, Inc.	Healthcare	ZBH
3M Company	Industrials	MMM
Alaska Air Group, Inc.	Industrials	ALK
AMETEK, Inc.	Industrials	AME
A. O. Smith Corporation	Industrials	AOS
Automatic Data Processing, Inc.	Industrials	ADP
Avery Dennison Corporation	Industrials	AVY
The Boeing Company	Industrials	BA
C.H. Robinson Worldwide, Inc.	Industrials	CHRW
Caterpillar Inc.	Industrials	CAT
Cintas Corporation	Industrials	CTAS
CSX Corporation	Industrials	CSX
Cummins Inc.	Industrials	CMI
Deere & Company	Industrials	DE
Dover Corporation	Industrials	DOV
Eaton Corporation plc	Industrials	ETN
Emerson Electric Co.	Industrials	EMR
Equifax Inc.	Industrials	EFX

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Expeditors International of Washington, Inc.	Industrials	EXPD
Fastenal Company	Industrials	FAST
FedEx Corporation	Industrials	FDX
General Dynamics Corporation	Industrials	GD
General Electric Company	Industrials	GE
Global Payments Inc.	Industrials	GPN
W.W. Grainger, Inc.	Industrials	GWW
Honeywell International Inc.	Industrials	HON
IDEX Corporation	Industrials	IEX
Illinois Tool Works Inc.	Industrials	ITW
Jacobs Engineering Group Inc.	Industrials	J
J.B. Hunt Transport Services, Inc.	Industrials	JBHT
Johnson Controls International plc	Industrials	JCI
L3Harris Technologies, Inc.	Industrials	LHX
Lockheed Martin Corporation	Industrials	LMT
Masco Corporation	Industrials	MAS
Norfolk Southern Corporation	Industrials	NSC
Northrop Grumman Corporation	Industrials	NOC
Old Dominion Freight Line, Inc.	Industrials	ODFL
PACCAR Inc	Industrials	PCAR
Parker-Hanni n Corporation	Industrials	PH
Paychex, Inc.	Industrials	PAYX
Pentair plc	Industrials	PNR
Quanta Services, Inc.	Industrials	PWR
Raytheon Technologies Corporation	Industrials	RTX
Republic Services, Inc.	Industrials	RSG
Robert Half International Inc.	Industrials	RHI
Rockwell Automation, Inc.	Industrials	ROK
Roper Technologies, Inc.	Industrials	ROP
Snap-on Incorporated	Industrials	SNA
Southwest Airlines Co.	Industrials	LUV
Stanley Black & Decker, Inc.	Industrials	SWK
Textron Inc.	Industrials	TXT
Trane Technologies plc	Industrials	TT
Union Paci c Corporation	Industrials	UNP
United Parcel Service, Inc.	Industrials	UPS
United Rentals, Inc.	Industrials	URI
Westinghouse Air Brake Technologies Corporation	Industrials	WAB
Waste Management, Inc.	Industrials	WM
Alexandria Real Estate Equities, Inc.	Real Estate	ARE
American Tower Corporation	Real Estate	AMT
AvalonBay Communities, Inc.	Real Estate	AVB
Boston Properties, Inc.	Real Estate	BXP
Crown Castle International Corp.	Real Estate	CCI

Table B.1 continued from previous page

Duke Realty Corporation	Real Estate	DRE
Equinix, Inc.	Real Estate	EQIX
Equity Residential	Real Estate	EQR
Essex Property Trust, Inc.	Real Estate	ESS
Federal Realty Investment Trust	Real Estate	FRT
Healthpeak Properties, Inc.	Real Estate	PEAK
Host Hotels & Resorts, Inc.	Real Estate	HST
Iron Mountain Incorporated	Real Estate	IRM
Kimco Realty Corporation	Real Estate	KIM
Mid-America Apartment Communities, Inc.	Real Estate	MAA
Prologis	Real Estate	PLD
Public Storage	Real Estate	PSA
Realty Income Corporation	Real Estate	O
Regency Centers Corporation	Real Estate	REG
SBA Communications Corporation	Real Estate	SBAC
Simon Property Group, Inc.	Real Estate	SPG
UDR, Inc.	Real Estate	UDR
Ventas, Inc.	Real Estate	VTR
Vornado Realty Trust	Real Estate	VNO
Welltower Inc.	Real Estate	WELL
Weyerhaeuser Company	Real Estate	WY
Accenture plc	Technology	ACN
Adobe Inc.	Technology	ADBE
Advanced Micro Devices, Inc.	Technology	AMD
Akamai Technologies, Inc.	Technology	AKAM
Amphenol Corporation	Technology	APH
Analog Devices, Inc.	Technology	ADI
Ansys	Technology	ANSS
Apple Inc.	Technology	AAPL
Applied Materials, Inc.	Technology	AMAT
Autodesk, Inc.	Technology	ADSK
Cadence Design Systems, Inc.	Technology	CDNS
Cisco Systems, Inc.	Technology	CSCO
Citrix Systems, Inc.	Technology	CTXS
Cognizant Technology Solutions Corporation	Technology	CTSH
Corning Incorporated	Technology	GLW
DXC Technology Company	Technology	DXC
F5, Inc.	Technology	FFIV
Fidelity National Information Services, Inc.	Technology	FIS
Fiserv, Inc.	Technology	FISV
Garmin Ltd.	Technology	GRMN
Gartner, Inc.	Technology	IT
HP Inc.	Technology	HPQ
Intel Corporation	Technology	INTC

Table B.1 continued from previous page

International Business Machines Corporation	Technology	IBM
Intuit Inc.	Technology	INTU
Jack Henry & Associates, Inc.	Technology	JKHY
Juniper Networks, Inc.	Technology	JNPR
KLA Corporation	Technology	KLAC
Lam Research Corporation	Technology	LRCX
Microchip Technology Incorporated	Technology	MCHP
Micron Technology, Inc.	Technology	MU
Microsoft Corporation	Technology	MSFT
Motorola Solutions, Inc.	Technology	MSI
NetApp, Inc.	Technology	NTAP
NortonLifeLock Inc.	Technology	NLOK
NVIDIA Corporation	Technology	NVDA
Oracle Corporation	Technology	ORCL
PTC Inc.	Technology	PTC
QUALCOMM Incorporated	Technology	QCOM
Skyworks Solutions, Inc.	Technology	SWKS
Synopsys, Inc.	Technology	SNPS
Teledyne Technologies Incorporated	Technology	TDY
Teradyne, Inc.	Technology	TER
Texas Instruments Incorporated	Technology	TXN
Trimble Inc.	Technology	TRMB
Tyler Technologies, Inc.	Technology	TYL
VeriSign, Inc.	Technology	VRSN
Western Digital Corporation	Technology	WDC
Xilinx	Technology	XLNX
Zebra Technologies Corporation	Technology	ZBRA
The AES Corporation	Utilities	AES
Alliant Energy Corporation	Utilities	LNT
Ameren Corporation	Utilities	AEE
American Electric Power Company, Inc.	Utilities	AEP
Atmos Energy Corporation	Utilities	ATO
CenterPoint Energy, Inc.	Utilities	CNP
CMS Energy Corporation	Utilities	CMS
Consolidated Edison, Inc.	Utilities	ED
Dominion Energy, Inc.	Utilities	D
DTE Energy Company	Utilities	DTE
Duke Energy Corporation	Utilities	DUK
Edison International	Utilities	EIX
Entergy Corporation	Utilities	ETR
Evergy, Inc.	Utilities	EVRG
Eversource Energy	Utilities	ES
Exelon Corporation	Utilities	EXC
FirstEnergy Corp.	Utilities	FE

Table B.1 continued from previous page

NextEra Energy, Inc.	Utilities	NEE
NiSource Inc.	Utilities	NI
Pinnacle West Capital Corporation	Utilities	PNW
PPL Corporation	Utilities	PPL
Public Service Enterprise Group Incorporated	Utilities	PEG
Sempra	Utilities	SRE
The Southern Company	Utilities	SO
WEC Energy Group, Inc.	Utilities	WEC
Xcel Energy Inc.	Utilities	XEL

Table B.2: Description of the constituents that belong to the portfolio traded in the Japanese market. For each constituent, the respective company name, sector and symbol are given.

Company Name	Sector	Symbol
Toyobo Co., Ltd.	Basic Materials	3101.T
Unitika Ltd.	Basic Materials	3103.T
Oji Holdings Corporation	Basic Materials	3861.T
Asahi Kasei Corporation	Basic Materials	3407.T
Denka Company Limited	Basic Materials	4061.T
DIC Corporation	Basic Materials	4631.T
Kuraray Co., Ltd.	Basic Materials	3405.T
Mitsubishi Chemical Holdings Corporation	Basic Materials	4188.T
Mitsui Chemicals, Inc.	Basic Materials	4183.T
Nissan Chemical Corporation	Basic Materials	4021.T
Nitto Denko Corporation	Basic Materials	6988.T
Shin-Etsu Chemical Co., Ltd.	Basic Materials	4063.T
Showa Denko K.K.	Basic Materials	4004.T
Sumitomo Chemical Company, Limited	Basic Materials	4005.T
Tokuyama Corporation	Basic Materials	4043.T
Tosoh Corporation	Basic Materials	4042.T
UBE Corporation	Basic Materials	4208.T
AGC Inc.	Basic Materials	5201.T
Sumitomo Osaka Cement Co., Ltd.	Basic Materials	5232.T
Taiheiyo Cement Corporation	Basic Materials	5233.T
Tokai Carbon Co., Ltd.	Basic Materials	5301.T
Kobe Steel, Ltd.	Basic Materials	5406.T
Nippon Steel Corporation	Basic Materials	5401.T
Paci c Metals Co., Ltd.	Basic Materials	5541.T
Dowa Holdings Co., Ltd.	Basic Materials	5714.T
Mitsubishi Materials Corporation	Basic Materials	5711.T
Nippon Light Metal Holdings Company, Ltd.	Basic Materials	5703.T
Sumitomo Metal Mining Co., Ltd.	Basic Materials	5713.T
Toho Zinc Co., Ltd.	Basic Materials	5707.T
KDDI Corporation	Communication Services	9433.T

Table B.2 continued from previous page

Nippon Telegraph and Telephone Corporation	Communication Services	9432.T
SoftBank Group Corp.	Communication Services	9984.T
CyberAgent, Inc.	Communication Services	4751.T
Dentsu Group Inc.	Communication Services	4324.T
Konami Holdings Corporation	Communication Services	9766.T
Nintendo Co., Ltd.	Communication Services	7974.T
Toho Co., Ltd.	Communication Services	9602.T
HASEKO Corporation	Consumer Cyclical	1808.T
Sekisui House, Ltd.	Consumer Cyclical	1928.T
Toray Industries, Inc.	Consumer Cyclical	3402.T
Bridgestone Corporation	Consumer Cyclical	5108.T
The Yokohama Rubber Co., Ltd.	Consumer Cyclical	5101.T
Nippon Sheet Glass Company, Limited	Consumer Cyclical	5202.T
Sumitomo Electric Industries, Ltd.	Consumer Cyclical	5802.T
JTEKT Corporation	Consumer Cyclical	6473.T
NSK Ltd.	Consumer Cyclical	6471.T
DENSO Corporation	Consumer Cyclical	6902.T
Hino Motors, Ltd.	Consumer Cyclical	7205.T
Honda Motor Co., Ltd.	Consumer Cyclical	7267.T
Isuzu Motors Limited	Consumer Cyclical	7202.T
Mazda Motor Corporation	Consumer Cyclical	7261.T
Mitsubishi Motors Corporation	Consumer Cyclical	7211.T
Nissan Motor Co., Ltd.	Consumer Cyclical	7201.T
Subaru Corporation	Consumer Cyclical	7270.T
Suzuki Motor Corporation	Consumer Cyclical	7269.T
Toyota Motor Corporation	Consumer Cyclical	7203.T
Yamaha Motor Co., Ltd.	Consumer Cyclical	7272.T
Nikon Corporation	Consumer Cyclical	7731.T
BANDAI NAMCO Holdings Inc.	Consumer Cyclical	7832.T
Yamaha Corporation	Consumer Cyclical	7951.T
Aeon Co., Ltd.	Consumer Cyclical	8267.T
Fast Retailing Co., Ltd.	Consumer Cyclical	9983.T
Isetan Mitsukoshi Holdings Ltd.	Consumer Cyclical	3099.T
J. Front Retailing Co., Ltd.	Consumer Cyclical	3086.T
Takashimaya Company, Limited	Consumer Cyclical	8233.T
Tokyu Corporation	Consumer Cyclical	9005.T
Rakuten Group, Inc.	Consumer Cyclical	4755.T
Nippon Suisan Kaisha, Ltd.	Consumer Defensive	1332.T
Ajinomoto Co., Inc.	Consumer Defensive	2802.T
Asahi Group Holdings, Ltd.	Consumer Defensive	2502.T
Japan Tobacco Inc.	Consumer Defensive	2914.T
Kikkoman Corporation	Consumer Defensive	2801.T
Kirin Holdings Company, Limited	Consumer Defensive	2503.T
Meiji Holdings Co., Ltd.	Consumer Defensive	2269.T

Table B.2 continued from previous page

Nichirei Corporation	Consumer Defensive	2871.T
NH Foods Ltd.	Consumer Defensive	2282.T
Nisshin Seifun Group Inc.	Consumer Defensive	2002.T
Sapporo Holdings Limited	Consumer Defensive	2501.T
Takara Holdings Inc.	Consumer Defensive	2531.T
Kao Corporation	Consumer Defensive	4452.T
Shiseido Company, Limited	Consumer Defensive	4911.T
Seven & i Holdings Co., Ltd.	Consumer Defensive	3382.T
Inpex Corporation	Energy	1605.T
ENEOS Holdings, Inc.	Energy	5020.T
Marui Group Co., Ltd.	Financial Services	8252.T
The Chiba Bank, Ltd.	Financial Services	8331.T
Sumitomo Mitsui Trust Holdings, Inc.	Financial Services	8309.T
Fukuoka Financial Group, Inc.	Financial Services	8354.T
Resona Holdings, Inc.	Financial Services	8308.T
The Shizuoka Bank, Ltd.	Financial Services	8355.T
Sumitomo Mitsui Financial Group, Inc.	Financial Services	8316.T
Daiwa Securities Group Inc.	Financial Services	8601.T
Matsui Securities Co., Ltd.	Financial Services	8628.T
Nomura Holdings, Inc.	Financial Services	8604.T
Credit Saison Co., Ltd.	Financial Services	8253.T
Astellas Pharma Inc.	Healthcare	4503.T
Chugai Pharmaceutical Co., Ltd.	Healthcare	4519.T
Daiichi Sankyo Company, Limited	Healthcare	4568.T
Sumitomo Pharma Co., Ltd.	Healthcare	4506.T
Eisai Co., Ltd.	Healthcare	4523.T
Kyowa Kirin Co., Ltd.	Healthcare	4151.T
Shionogi & Co., Ltd.	Healthcare	4507.T
Takeda Pharmaceutical Company Limited	Healthcare	4502.T
Olympus Corporation	Healthcare	7733.T
Terumo Corporation	Healthcare	4543.T
JGC Holdings Corporation	Industrials	1963.T
Kajima Corporation	Industrials	1812.T
Obayashi Corporation	Industrials	1802.T
Shimizu Corporation	Industrials	1803.T
Taisei Corporation	Industrials	1801.T
Teijin Limited	Industrials	3401.T
FUJIFILM Holdings Corporation	Industrials	4901.T
NGK Insulators, Ltd.	Industrials	5333.T
Toto Ltd.	Industrials	5332.T
Fujikura Ltd.	Industrials	5803.T
Furukawa Electric Co., Ltd.	Industrials	5801.T
Mitsui Mining & Smelting Co., Ltd.	Industrials	5706.T
Amada Co., Ltd.	Industrials	6113.T

Table B.2 continued from previous page

Daikin Industries,Ltd.	Industrials	6367.T
Ebara Corporation	Industrials	6361.T
Hitachi Construction Machinery Co., Ltd.	Industrials	6305.T
Hitachi Zosen Corporation	Industrials	7004.T
IHI Corporation	Industrials	7013.T
The Japan Steel Works, Ltd.	Industrials	5631.T
Komatsu Ltd.	Industrials	6301.T
Kubota Corporation	Industrials	6326.T
Mitsubishi Heavy Industries, Ltd.	Industrials	7011.T
NTN Corporation	Industrials	6472.T
Okuma Corporation	Industrials	6103.T
Sumitomo Heavy Industries, Ltd.	Industrials	6302.T
Fanuc Corporation	Industrials	6954.T
Fuji Electric Co., Ltd.	Industrials	6504.T
GS Yuasa Corporation	Industrials	6674.T
Hitachi, Ltd.	Industrials	6501.T
Mitsubishi Electric Corporation	Industrials	6503.T
Ricoh Company, Ltd.	Industrials	7752.T
YASKAWA Electric Corporation	Industrials	6506.T
Yokogawa Electric Corporation	Industrials	6841.T
Mitsui E&S Holdings Co., Ltd.	Industrials	7003.T
Citizen Watch Co., Ltd.	Industrials	7762.T
Konica Minolta, Inc.	Industrials	4902.T
ITOCHU Corporation	Industrials	8001.T
Marubeni Corporation	Industrials	8002.T
Mitsubishi Corporation	Industrials	8058.T
Mitsui & Co., Ltd.	Industrials	8031.T
Sojitz Corporation	Industrials	2768.T
Sumitomo Corporation	Industrials	8053.T
Toyota Tsusho Corporation	Industrials	8015.T
Dai Nippon Printing Co., Ltd.	Industrials	7912.T
Toppan Inc.	Industrials	7911.T
Central Japan Railway Company	Industrials	9022.T
East Japan Railway Company	Industrials	9020.T
Keio Corporation	Industrials	9008.T
Keisei Electric Railway Co., Ltd.	Industrials	9009.T
Odakyu Electric Railway Co., Ltd.	Industrials	9007.T
Tobu Railway Co., Ltd.	Industrials	9001.T
West Japan Railway Company	Industrials	9021.T
Nippon Express Co., Ltd.	Industrials	9062.T
Yamato Holdings Co., Ltd.	Industrials	9064.T
Kawasaki Kisen Kaisha, Ltd.	Industrials	9107.T
Mitsui O.S.K. Lines, Ltd.	Industrials	9104.T
ANA Holdings Inc.	Industrials	9202.T

Table B.2 continued from previous page

Mitsubishi Logistics Corporation	Industrials	9301.T
SECOM CO., LTD.	Industrials	9735.T
Daiwa House Industry Co., Ltd.	Real Estate	1925.T
Mitsubishi Estate Co., Ltd.	Real Estate	8802.T
Mitsui Fudosan Co., Ltd.	Real Estate	8801.T
Sumitomo Realty & Development Co., Ltd.	Real Estate	8830.T
Tokyo Tatemono Co., Ltd.	Real Estate	8804.T
Nippon Electric Glass Co., Ltd.	Technology	5214.T
Advantest Corporation	Technology	6857.T
Alps Alpine Co., Ltd.	Technology	6770.T
Canon Inc.	Technology	7751.T
Casio Computer Co.,Ltd.	Technology	6952.T
SCREEN Holdings Co., Ltd.	Technology	7735.T
Fujitsu Limited	Technology	6702.T
Keyence Corporation	Technology	6861.T
Kyocera Corporation	Technology	6971.T
MinebeaMitsumi Inc.	Technology	6479.T
Murata Manufacturing Co., Ltd.	Technology	6981.T
NEC Corporation	Technology	6701.T
Oki Electric Industry Co., Ltd.	Technology	6703.T
OMRON Corporation	Technology	6645.T
Panasonic Holdings Corporation	Technology	6752.T
Seiko Epson Corporation	Technology	6724.T
Sharp Corporation	Technology	6753.T
Sony Group Corporation	Technology	6758.T
Taiyo Yuden Co., Ltd.	Technology	6976.T
TDK Corporation	Technology	6762.T
Tokyo Electron Limited	Technology	8035.T
NTT DATA Corporation	Technology	9613.T
Trend Micro Incorporated	Technology	4704.T
Chubu Electric Power Company, Incorporated	Utilities	9502.T
The Kansai Electric Power Company, Incorporated	Utilities	9503.T
Tokyo Electric Power Company Holdings, Incorporated	Utilities	9501.T
Osaka Gas Co., Ltd.	Utilities	9532.T
Tokyo Gas Co.,Ltd.	Utilities	9531.T

Table B.3: Description of the constituents that belong to the portfolio traded in the European market. For each constituent, the respective company name, sector and symbol are given.

Company Name	Sector	Symbol
L'Air Liquide S.A.	Basic Materials	AI.PA
BASF SE	Basic Materials	BAS.DE
CRH plc	Basic Materials	CRG.IR
Deutsche Telekom AG	Communication Services	DTE.DE

Table B.3 continued from previous page

Bayerische Motoren Werke Aktiengesellschaft	Consumer Cyclical	BMW.DE
Daimler AG	Consumer Cyclical	DAI.DE
Stellantis N.V.	Consumer Cyclical	STLA.MI
Volkswagen AG	Consumer Cyclical	VOW.DE
Flutter Entertainment plc	Consumer Cyclical	FLTR.IR
Industria de Diseño Textil, S.A.	Consumer Cyclical	ITX.MC
Kering SA	Consumer Cyclical	KER.PA
LVMH Moët Hennessy - Louis Vuitton, Societe Europeenne	Consumer Cyclical	MC.PA
adidas AG	Consumer Cyclical	ADS.DE
Pernod Ricard SA	Consumer Defensive	RI.PA
Anheuser-Busch InBev SA/NV	Consumer Defensive	ABI.BR
Danone S.A.	Consumer Defensive	BN.PA
L'Oreal S.A.	Consumer Defensive	OR.PA
Eni S.p.A.	Energy	ENI.MI
TotalEnergies SE	Energy	TTE.PA
Banco Bilbao Vizcaya Argentaria, S.A.	Financial Services	BBVA.MC
Banco Santander, S.A.	Financial Services	SAN.MC
BNP Paribas SA	Financial Services	BNP.PA
ING Groep N.V.	Financial Services	INGA.AS
Intesa Sanpaolo S.p.A.	Financial Services	ISP.MI
Deutsche Börse AG	Financial Services	DB1.DE
Allianz SE	Financial Services	ALV.DE
AXA SA	Financial Services	CS.PA
Münchener Rückversicherungs	Financial Services	MUV2.DE
Bayer Aktiengesellschaft	Healthcare	BAYN.DE
Koninklijke Philips N.V.	Healthcare	PHIA.AS
EssilorLuxottica Societe anonyme	Healthcare	EL.PA
Sano	Healthcare	SAN.PA
Airbus SE	Industrials	AIR.PA
Safran SA	Industrials	SAF.PA
Vinci SA	Industrials	DG.PA
Schneider Electric S.E.	Industrials	SU.PA
Siemens Aktiengesellschaft	Industrials	SIE.DE
Deutsche Post AG	Industrials	DPW.DE
In neon Technologies AG	Technology	IFX.DE
ASML Holding N.V.	Technology	ASML.AS
SAP SE	Technology	SAP.DE
Enel SpA	Utilities	ENEL.MI
Iberdrola, S.A.	Utilities	IBE.MC

Table B.4: Description of the constituents that belong to the portfolio traded in the Hong Kong market. For each constituent, the respective company name, sector and symbol are given.

Company Name	Sector	Symbol
China Unicom (Hong Kong) Limited	Communication Services	0762.HK
China Mobile Limited	Communication Services	0941.HK
Galaxy Entertainment Group Limited	Consumer Cyclical	0027.HK
Hengan International Group Company Limited	Consumer Defensive	1044.HK
China Petroleum & Chemical Corporation	Energy	0386.HK
PetroChina Company Limited	Energy	0857.HK
HSBC Holdings plc	Financial Services	0005.HK
Hang Seng Bank Limited	Financial Services	0011.HK
Hong Kong Exchanges and Clearing Limited	Financial Services	0388.HK
CK Hutchison Holdings Limited	Industrials	0001.HK
Swire Paci c Limited	Industrials	0019.HK
MTR Corporation Limited	Industrials	0066.HK
CITIC Limited	Industrials	0267.HK
Henderson Land Development Company Limited	Real Estate	0012.HK
Sun Hung Kai Properties Limited	Real Estate	0016.HK
New World Development Company Limited	Real Estate	0017.HK
Sino Land Company Limited	Real Estate	0083.HK
Hang Lung Properties Limited	Real Estate	0101.HK
China Overseas Land & Investment Limited	Real Estate	0688.HK
China Resources Land Limited	Real Estate	1109.HK
CLP Holdings Limited	Utilities	0002.HK
The Hong Kong and China Gas Company Limited	Utilities	0003.HK
Power Assets Holdings Limited	Utilities	0006.HK
CK Infrastructure Holdings Limited	Utilities	1038.HK