Priority Queues

Priority Queues

Problem.

- Let S={(s1,p1), (s2,p2),…,(sn,pn)} where s(i) is a key and p(i) is the priority of s(i).

How to design a data structure/algorithm to support the following operations over S?

ExtractMin: Returns the element of S with minimum priority

Insert(s,p): Insert a new element (s,p) in S

RemoveMin: Remove the element in S with minimum p

Priority Queues

Solution 1. Used a sorted list

ExtractMin :
Insert:
DeleteMin:

Solution 2. Use a list with the pair with minimum p at the first position

ExtractMin :
Insert:
DeleteMin:

Can we do better? How?

Priority Queues

Solution 1. Used a sorted list

ExtractMin : O(1) time
Insert: O(n) time
DeleteMin: O(1) time

Solution 2. Use a list with the pair with minimum p at the first position

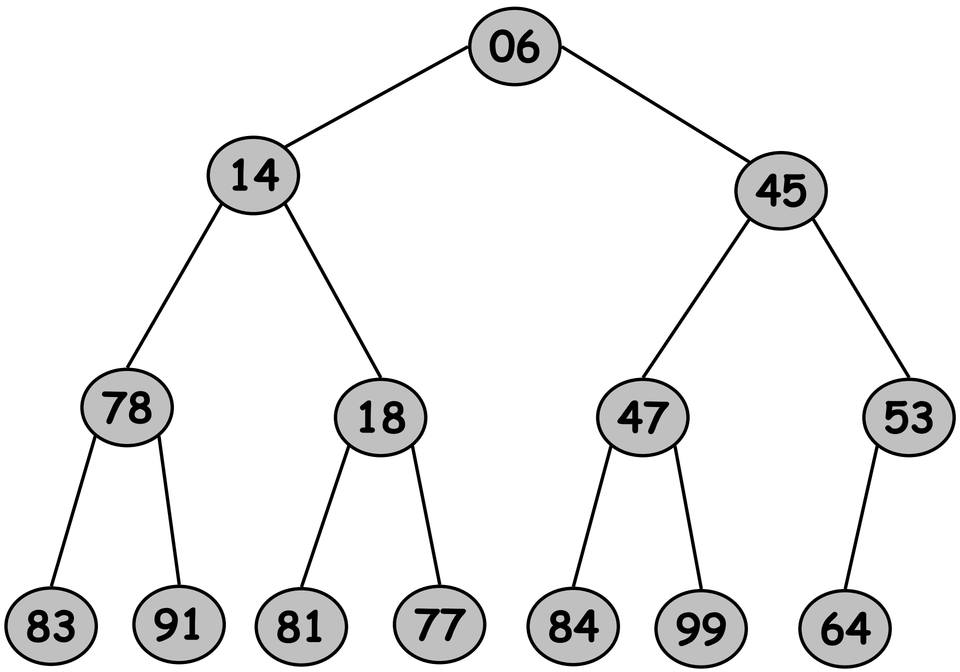
ExtractMin : O(1) time
Insert: O(1) time
DeleteMin: O(n) time

Can we do better? How?

Binary Heap: Definition

Binary heap.

- Almost complete binary tree
  - filled on all levels, except last, where filled from left to right
- Min-heap ordered
  - every child greater than (or equal to) parent



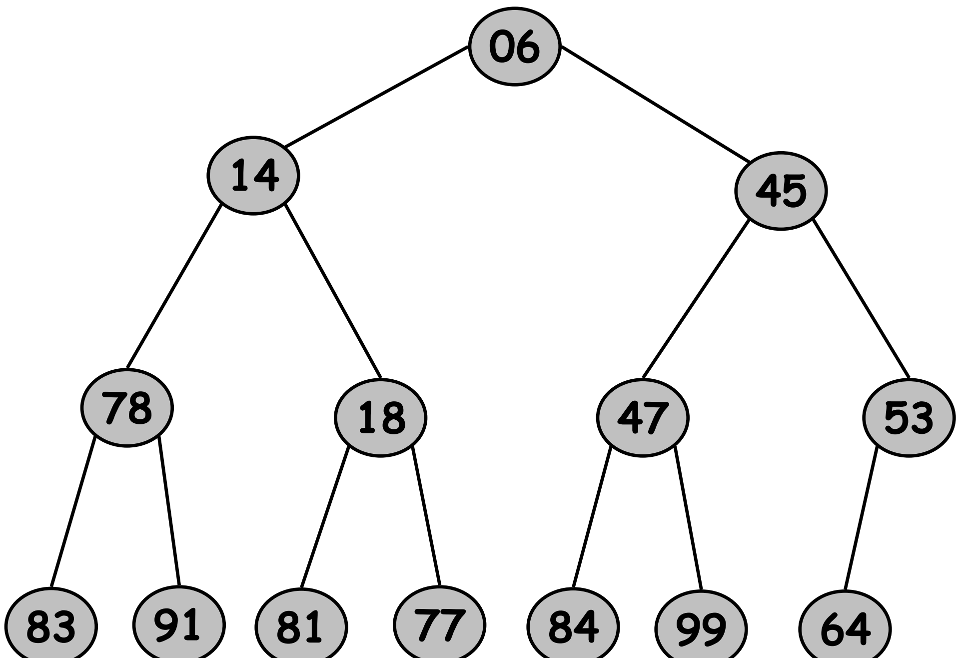
Max-heap is analogous (every child is smaller than its parent)

Binary Heap: Properties

Properties.

- Min element is in
- Heap with N elements has height

In particular can obtain minimum element (ExtractMin) in O(1)



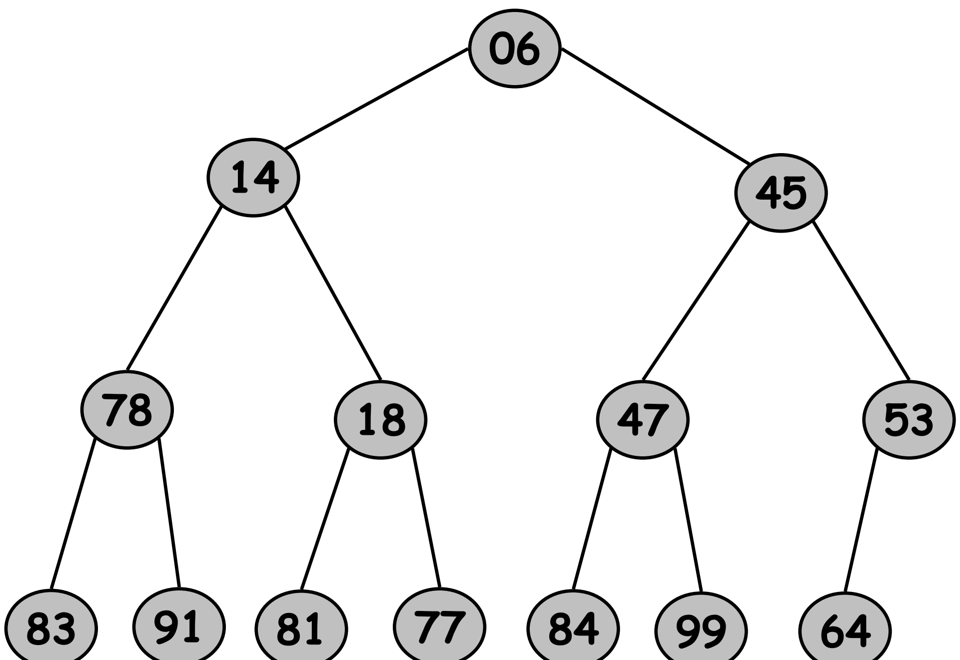
N = 14
Height = 3

Binary Heap: Properties

Properties.

- Min element is in root.
- Heap with N elements has height = $\lfloor \log_2 N \rfloor$.

In particular can obtain minimum element (ExtractMin) in O(1)



N = 14
Height = 3

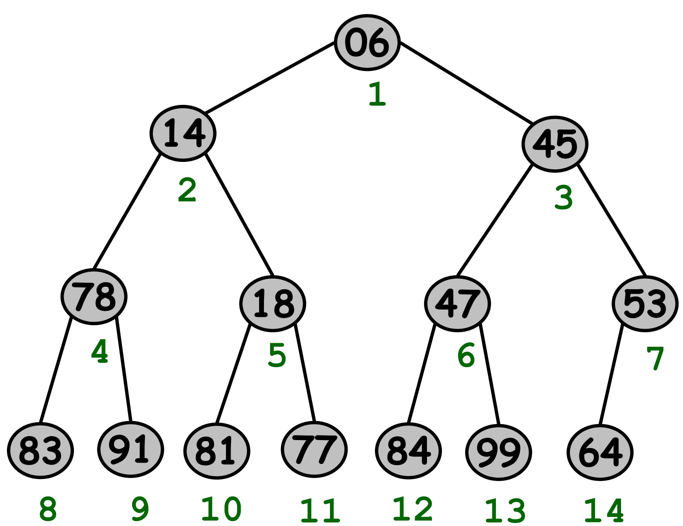
Binary Heaps: Array Implementation

Implementing binary heaps

- Use an array: no need for explicit parent or child pointers.
  - `Parent(i)` $= \lfloor i/2 \rfloor$
  - `Left(i)` $= 2i$
  - `Right(i)` $= 2i + 1$

Only important properties for us:

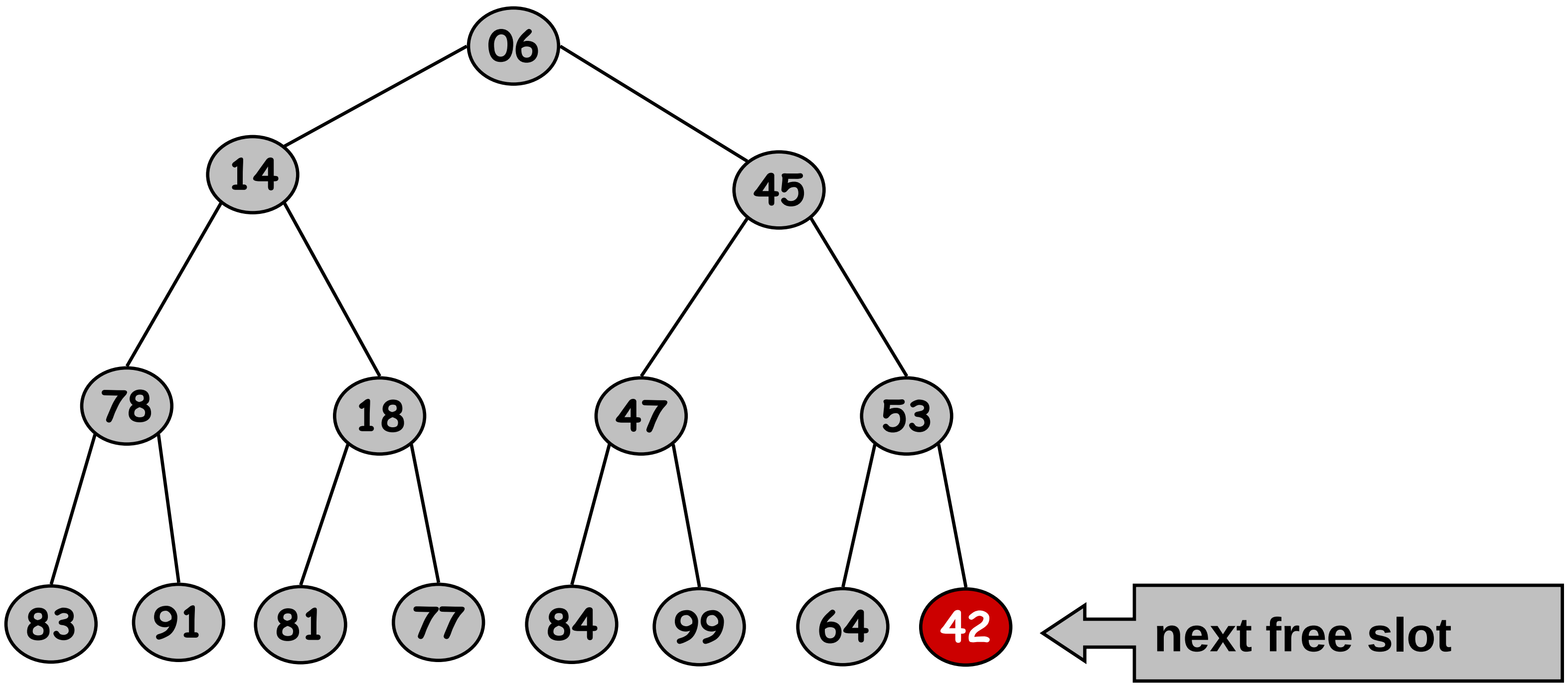
1. Last level of the tree is occupied from left to right
2. We can find the first (leftmost) empty space in constant time (keep track of first empty space with an integer firstEmpty)



Binary Heap: Insertion

Insert element x into heap.

- Insert into fist available slot.
- Bubble up until it's heap ordered.



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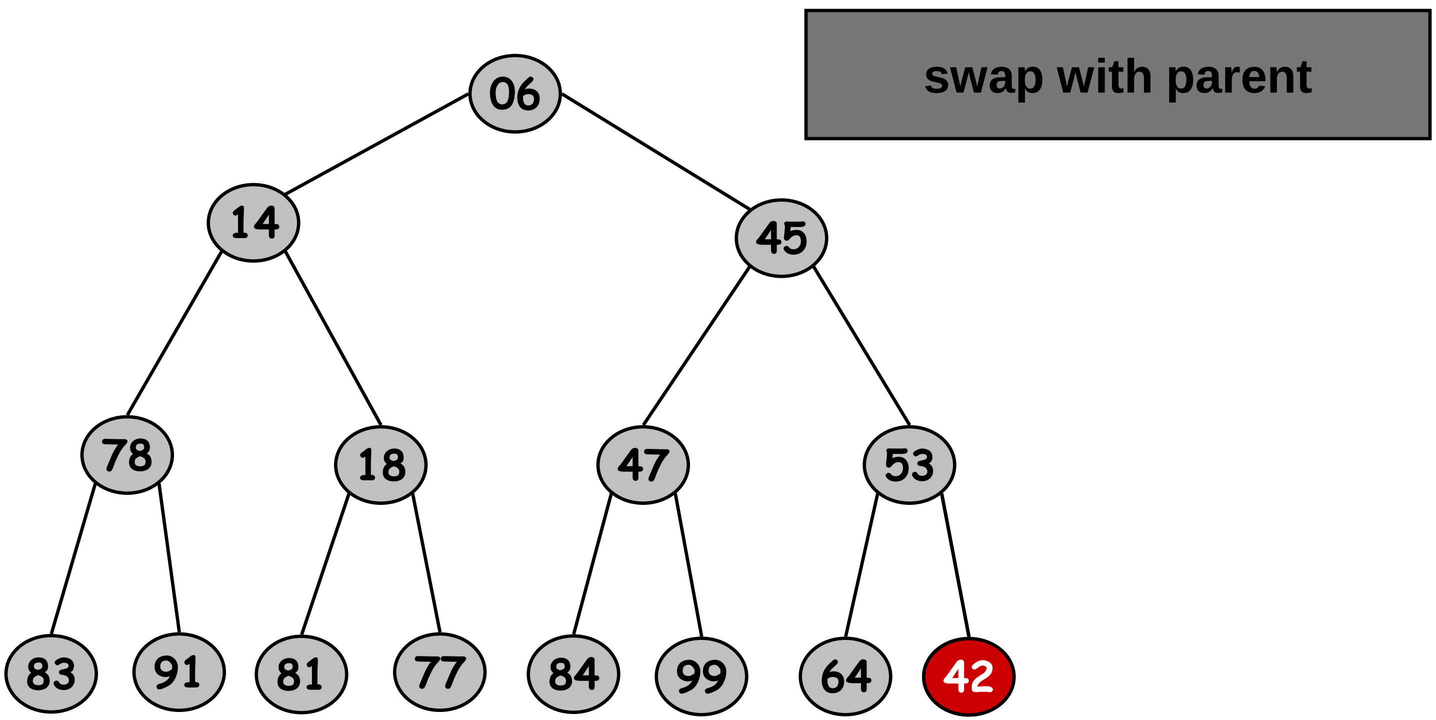

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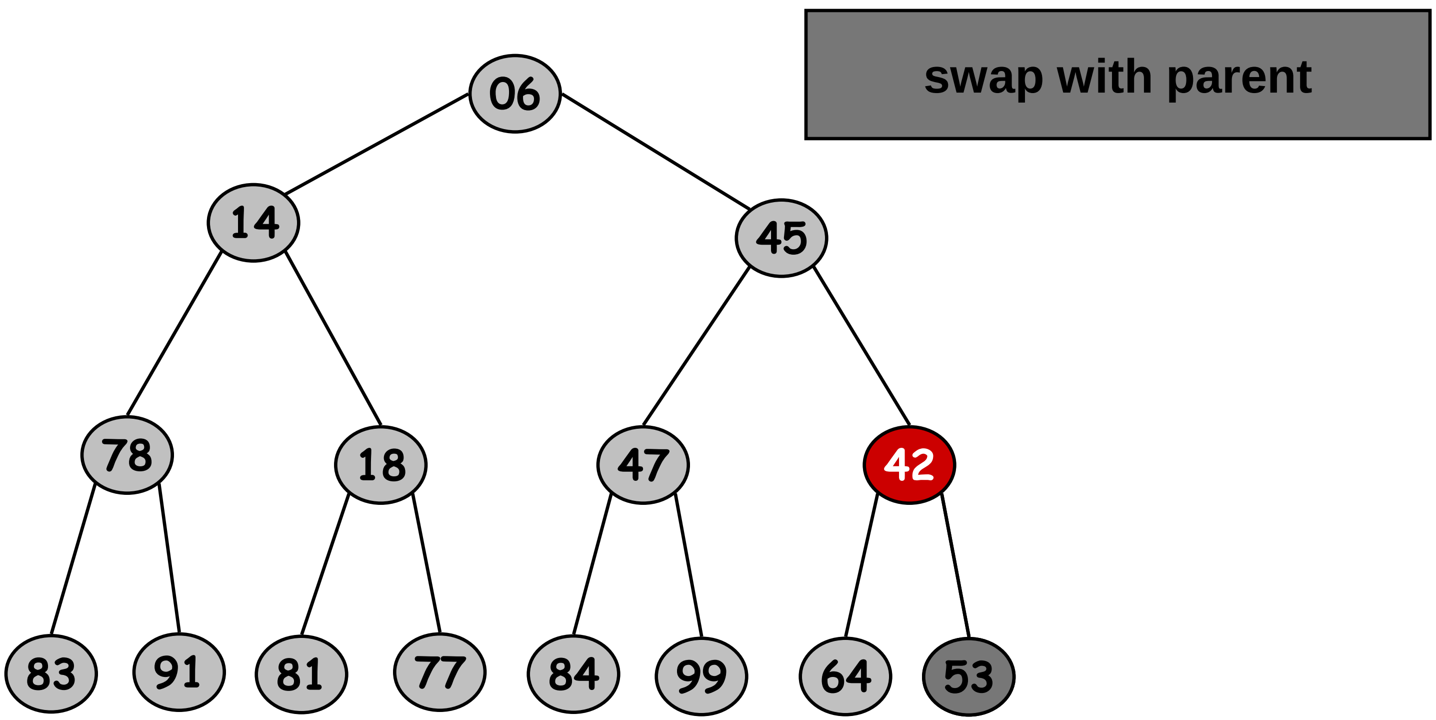
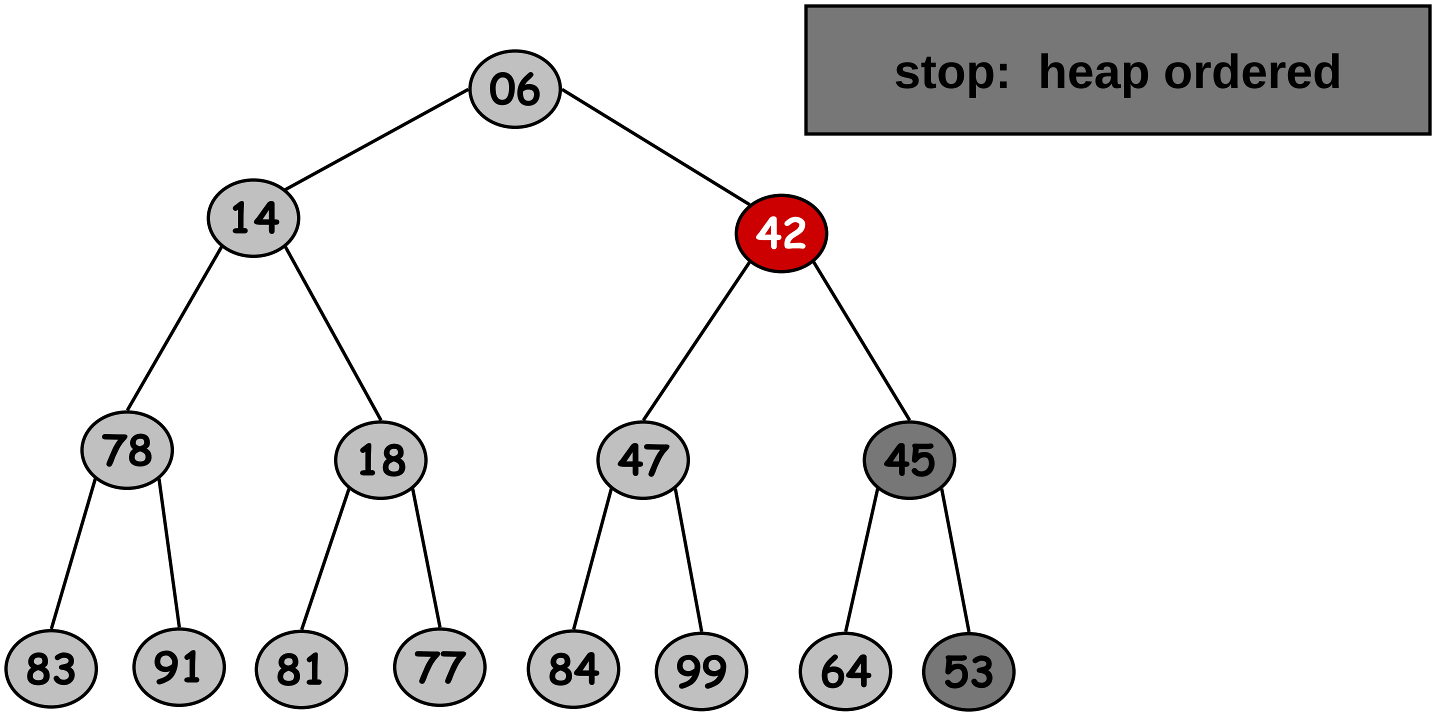


Binary Heap: Insertion

Insert element x into heap.
- Insert into first available slot.
- Bubble up until it's heap ordered.

Q: What is the time complexity of Insert?

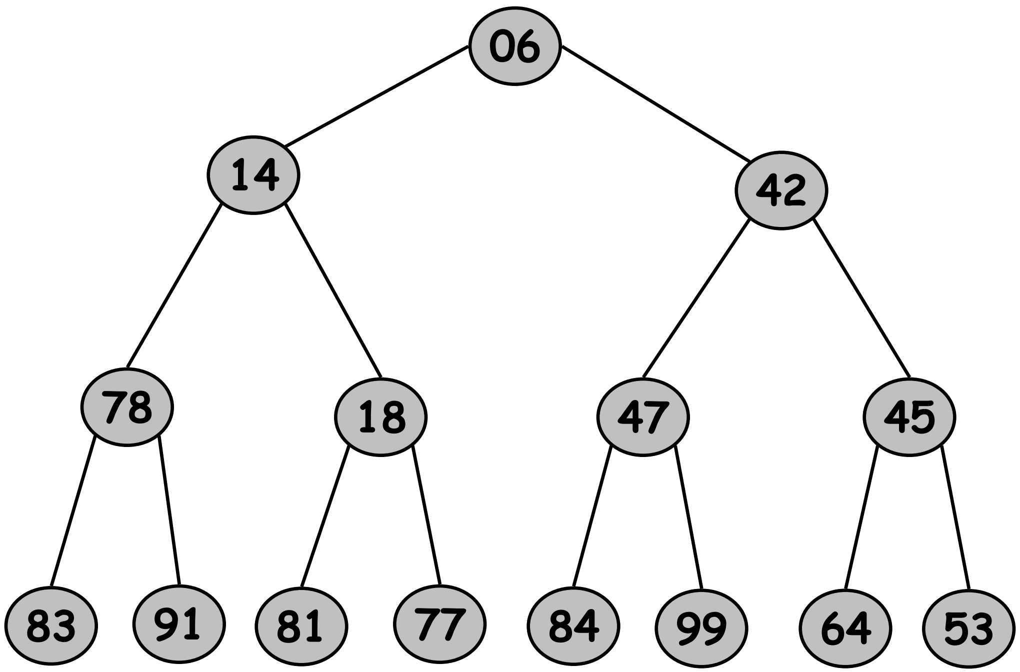
A: O(log N) operations



Binary Heap: Decrease Key

Decrease key of element x to k.

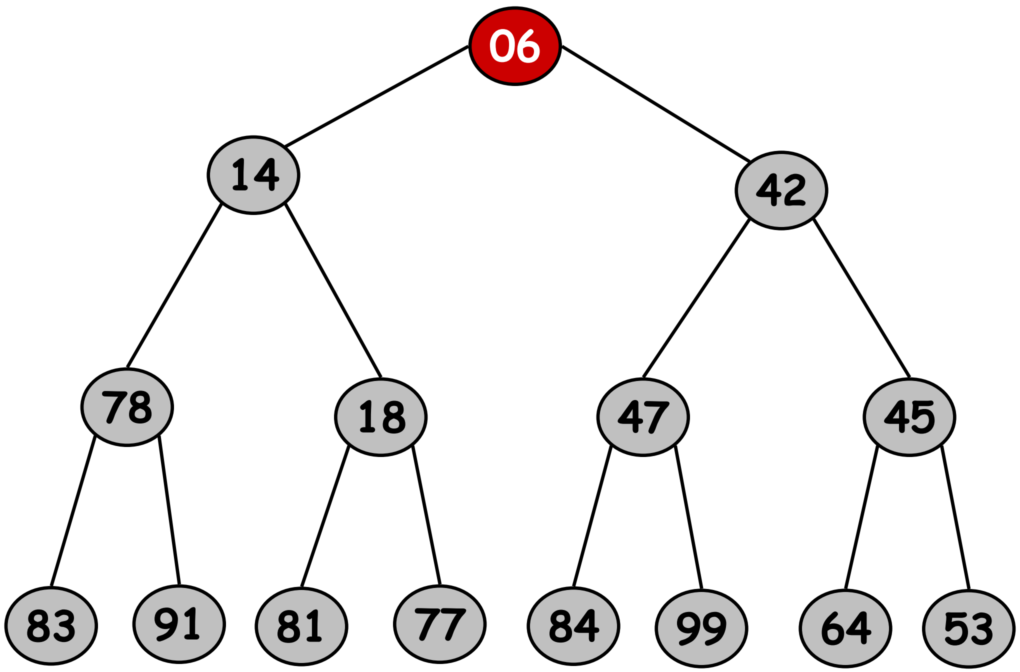
- Bubble up until it's heap ordered.
- O(log N) operations.



Binary Heap: Delete Min

Delete minimum element from heap.

- Exchange root with any leaf (rightmost one is easy to find)
- Delete this leaf
- Bubble root down until it's heap ordered, exchanging it with smallest child



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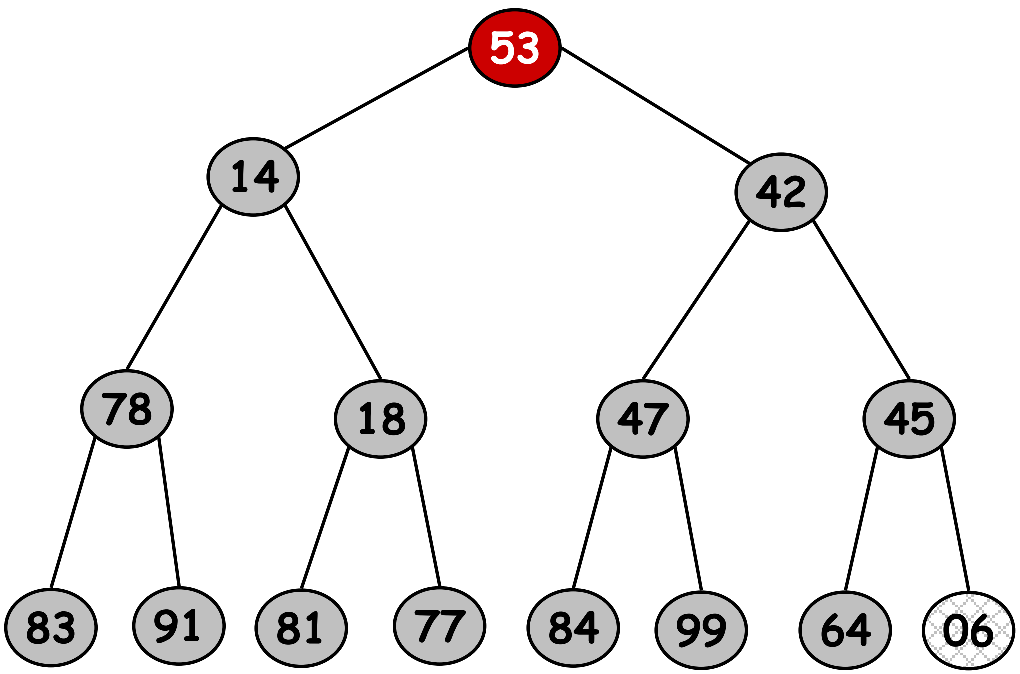

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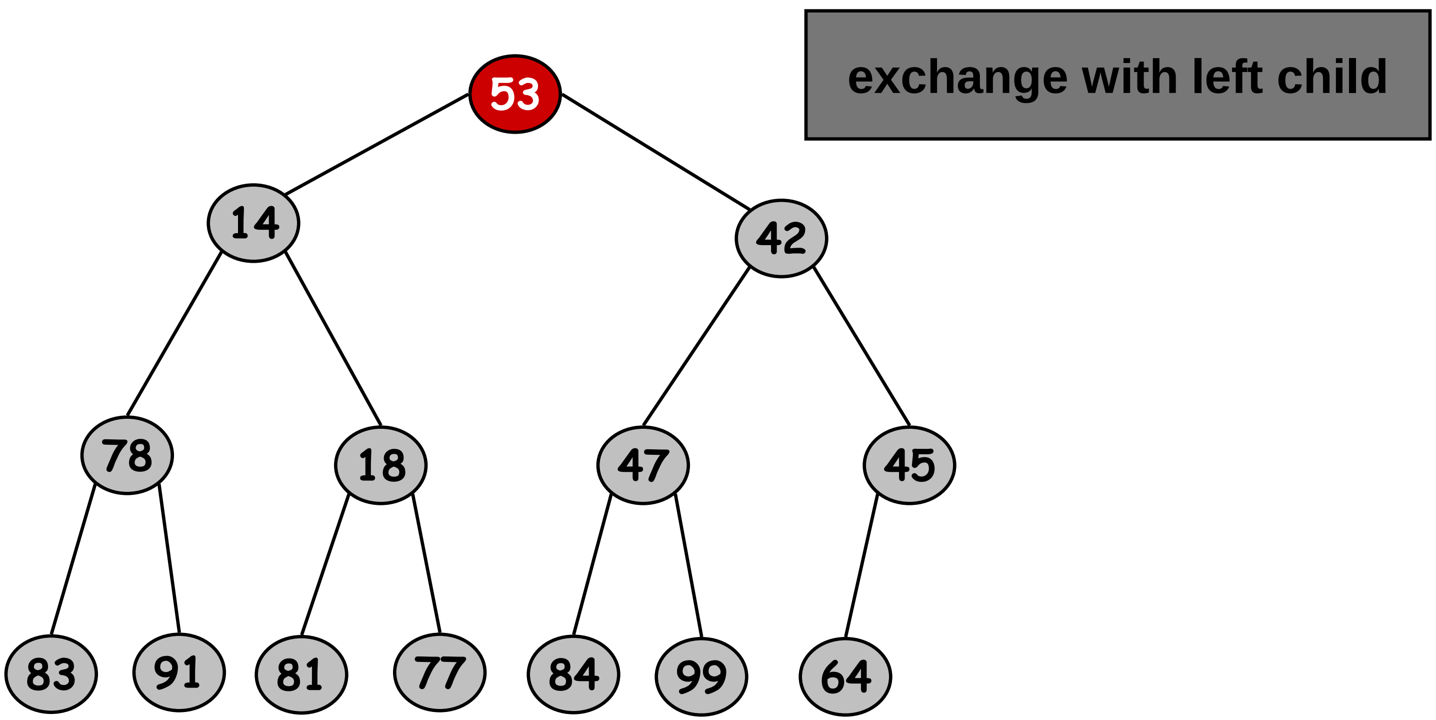

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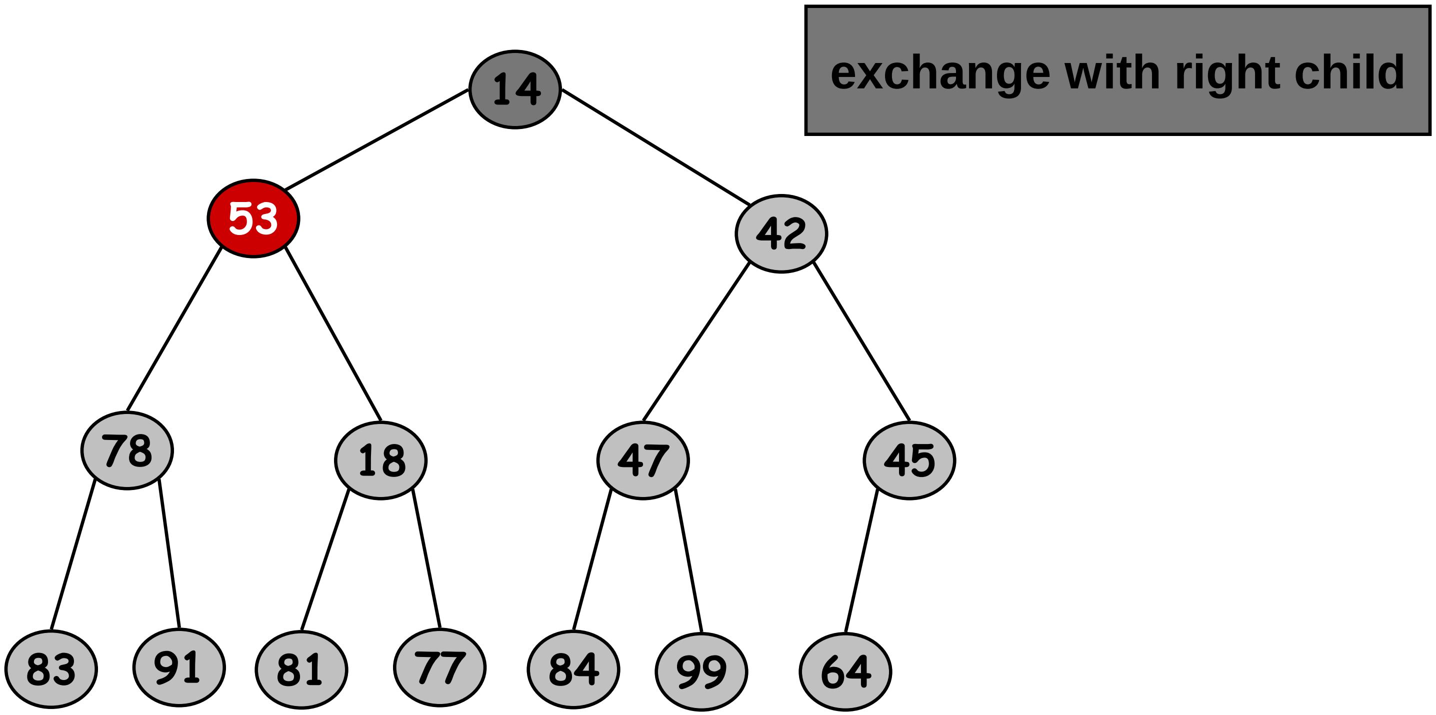

Binary Heap: Delete Min

Delete minimum element from heap.

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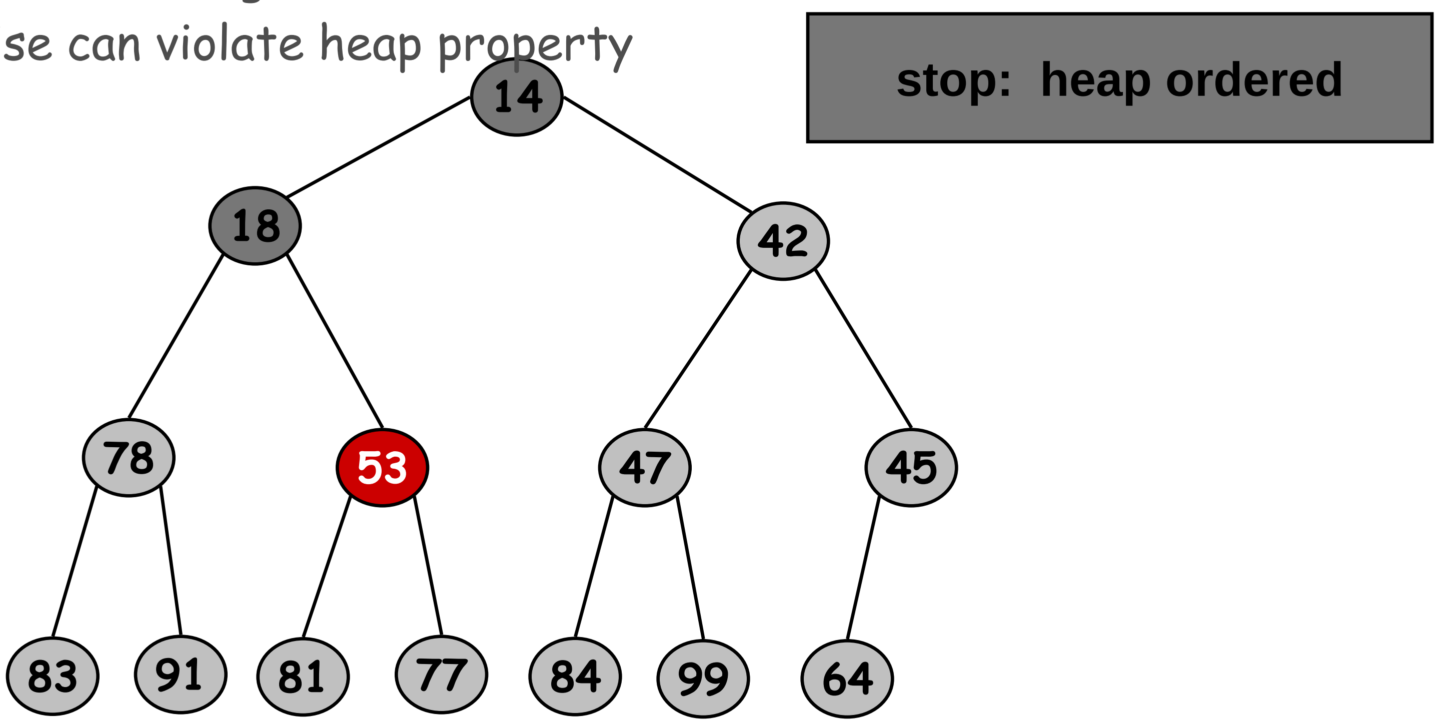
Q: Why do we exchange with smallest child?

A: Otherwise can violate heap property



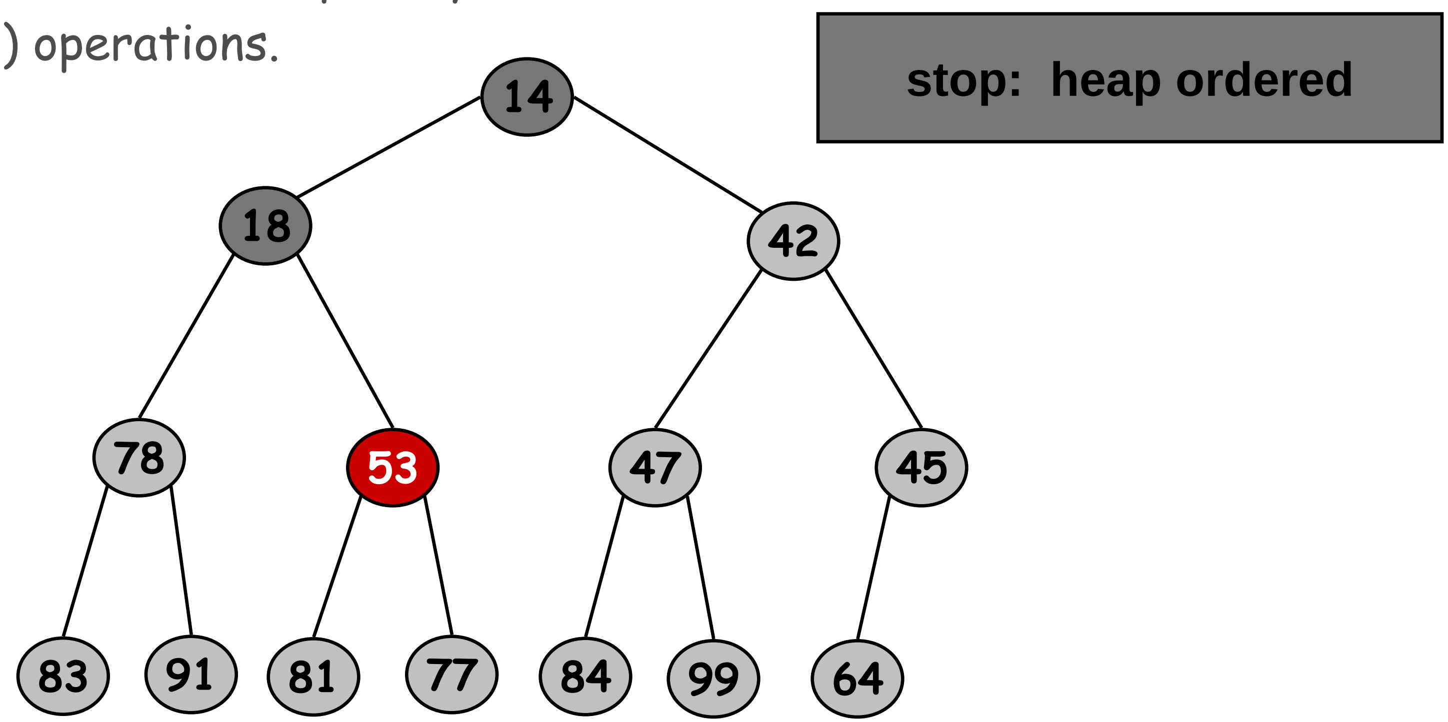
Binary Heap: Delete Min

Delete minimum element from heap.

- Exchange root with any leaf (rightmost one is easy to find)
- Delete this leaf
- Bubble root down until it's heap ordered, exchanging it with smallest child

Q: What is the time complexity of DeleteMin?

A: O(log N) operations.



Priority Queues

Solution 1. Use a sorted list
ExtractMin: O(1) time
Insert: O(n) time
DeleteMin: O(1) time

Solution 2. Use a list with the pair with minimum p at the first position
ExtractMin: O(1) time
Insert: O(1) time
DeleteMin: O(n) time

Solution 3. Binary Heap
ExtractMin: O(1) time
Insert: O(log n) time
DeleteMin: O(log n) time

Exercise

Exercise: Can we use use the procedures we know (Insert, ExtractMin, DeleteMin, DecreaseKey) to implement the IncreaseKey(x,k) that increases the key of an element x to value k?

If so, analyze the complexity of this new operation.

Solution: One possibility is to remove element x and add it with key k. To remove this value we can:

1) Obtain the minimum MIN using ExtractMin
2) reduce the key of x to MIN-1 using DecreaseKey (so now x is minimum)
3) Apply RemoveMin to remove it

The total complexity is O(log n), since it uses 4 operations that are at most O(log n).

Application: Heapsort

Q: Can we sort N numbers using a Binary Heap?

- Insert N items into a binary min-heap
  - O (N log N)
- Perform N delete-min operations.
  - O(N log N)
- Overall
  - O(N log N) sort.

Obs: If you use a max-heap can do with no extra storage (in-place)

Exercice

- Run the execution of Heapsort over list of numbers
  18 25 41 34 10 52 50 48

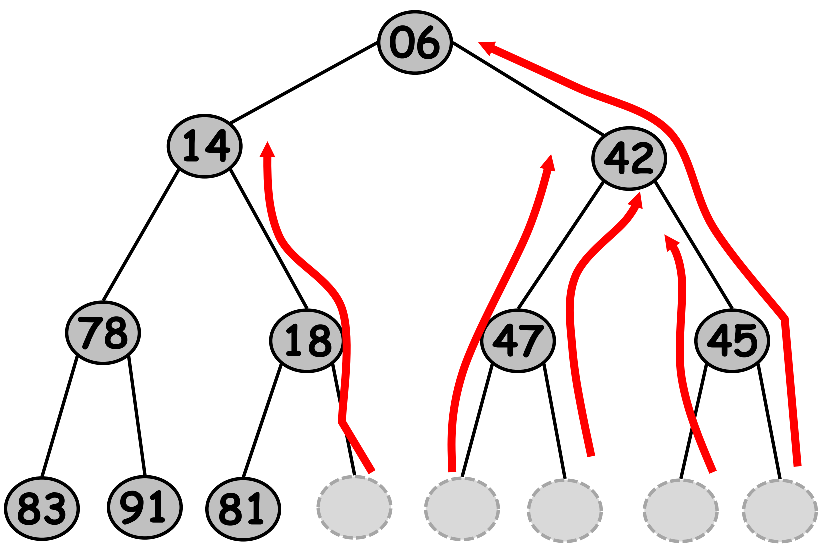
Building a Heap

In Heapsort, we just built a heap for n numbers by inserting them one at a time

Time complexity of this procedure: O(n log n)

Q: Can we build a heap faster?

Bottleneck: When adding the last few nodes, each may move a lot up in the heap, costs too much



Building a Heap in O(n)

Idea: Move nodes down instead of up

Operation down(node): keeps exchanging node with smallest child until it is in the right position, that is, has smaller priorities than its children

Building a Heap of numbers a1, a2, … an in O(n):

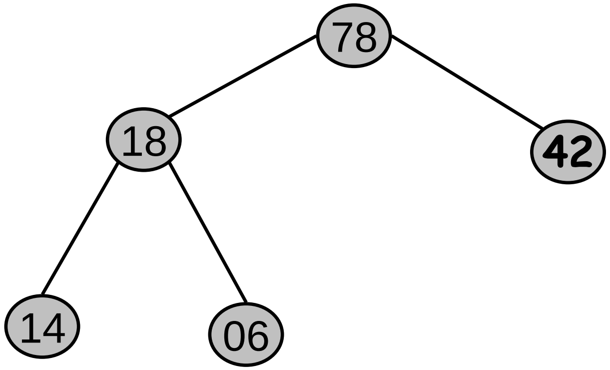
HeapBuild1 (Cormen)

- Make an almost complete binary tree with numbers in any position
- Traverse tree from bottom up applying down(node)

Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

Apply *down()* to node with 06



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

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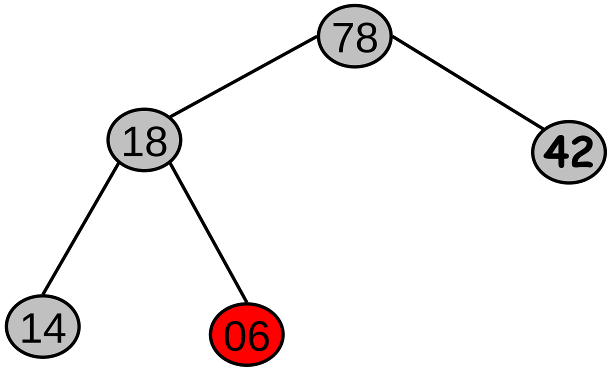
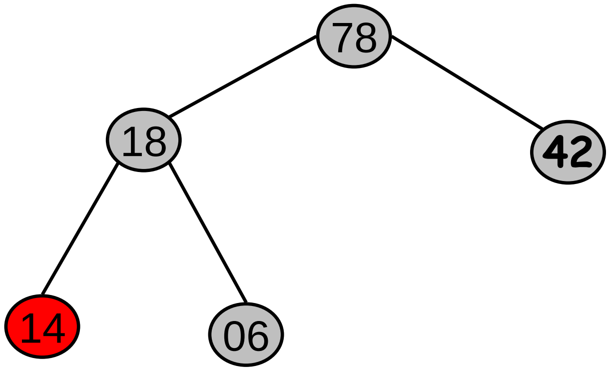


Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

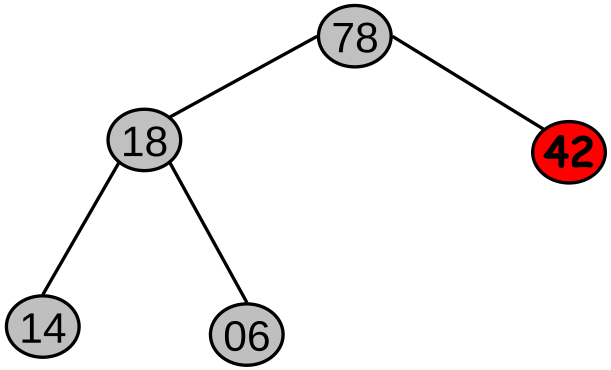
Apply *down()* to node with 14



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

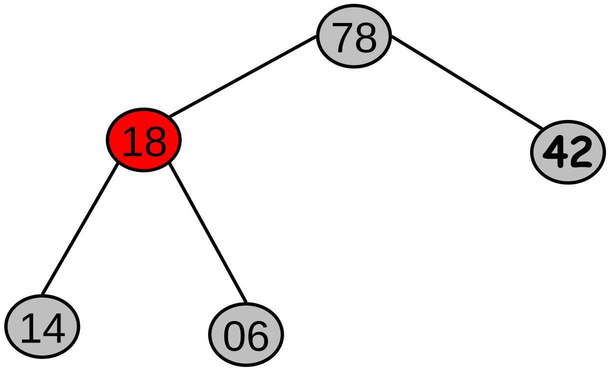
Apply *down()* to node with 42



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

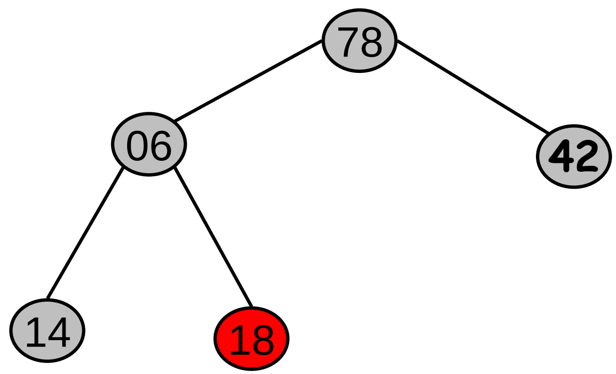
Apply *down()* to node with 18



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

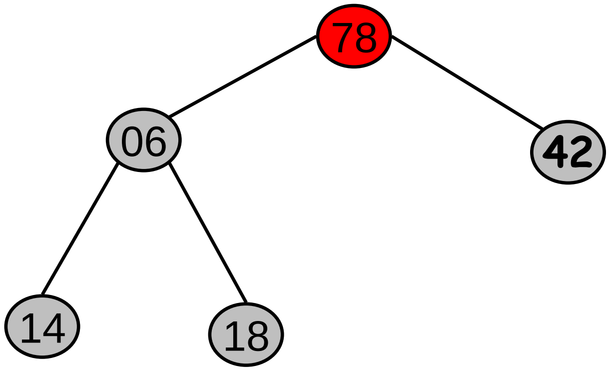
Apply *down()* to node with 18



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

Apply *down()* to node with 78 twice



Building a Heap in O(n)

Example: Numbers 14, 06, 42, 78, 18

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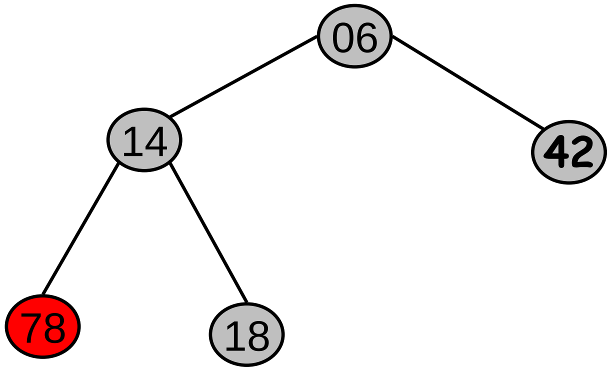

Building a Heap in O(n): Analysis

Q: Quantos nós temos no nível i de baixo pra cima?

- **Observação:** Para i>0, no i-ésimo nível do heap de baixo para cima temos no máximo $n / 2^{i-1}$ nós
  - O heap tem = $\lfloor \log_2 n \rfloor$ níveis. No nível 0, de baixo para cima, temos no mínimo 1 e no máximo N nós.
  - Todos os niveis excetos os dois últimos tem metade dos nós do nível imediatamente abaixo

Building a Heap in O(n): Analysis

Custo do down(node): se node está no nível i de baixo para cima, esse down custa O(i) operações

$$\mathit{Custo}\ \text{total} = \sum_{i=1}^{\log n} i.\frac{n}{2^i} =$$

Building a Heap in O(n): Analysis

Custo do down(node): se node está no nível i de baixo para cima, esse down custa O(i) operações

$$Custo \text{ total } = \sum_{i=1}^{\log n} i.\frac{n}{2^i} = \Theta(n)$$

Exercise

Exercicio: Seja S um conjunto de n numeros reais distintos. Explique como seria um algoritmo eficiente para encontrar os $\sqrt{n}$ menores numeros do conjunto S e analise sua complexidade. Quanto mais eficiente o algoritmo melhor.

Solucao: Crie uma min-heap com os elementos de S e aplique ExtractMin $\sqrt{n}$ vezes; retorne esses elementos removidos

Complexidade: O(n) pra montar o heap, $O(\sqrt{n}\,.\log(n)) = O(n)$ para remover of elementos => total: O(n)