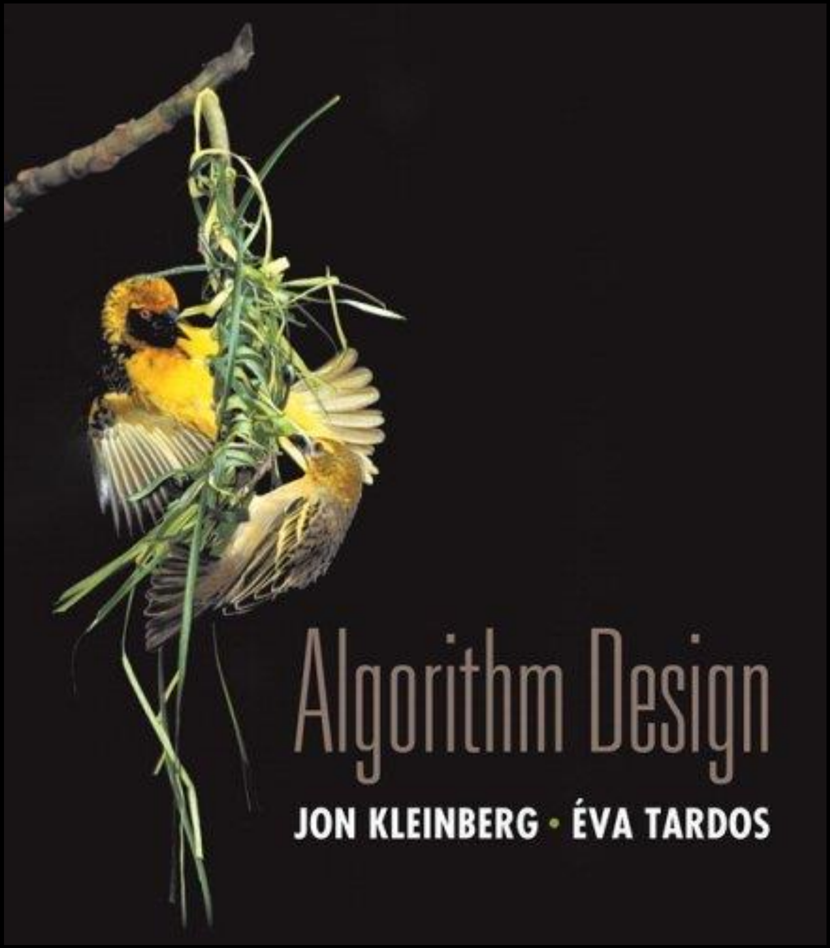


Chapter 5

Divide and Conquer

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}n$.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: $n \log n$.

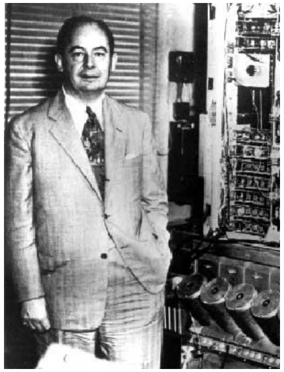
> Divide et impera.
> Veni, vidi, vici.
> \- *Julius Caesar*

5.1 Mergesort

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

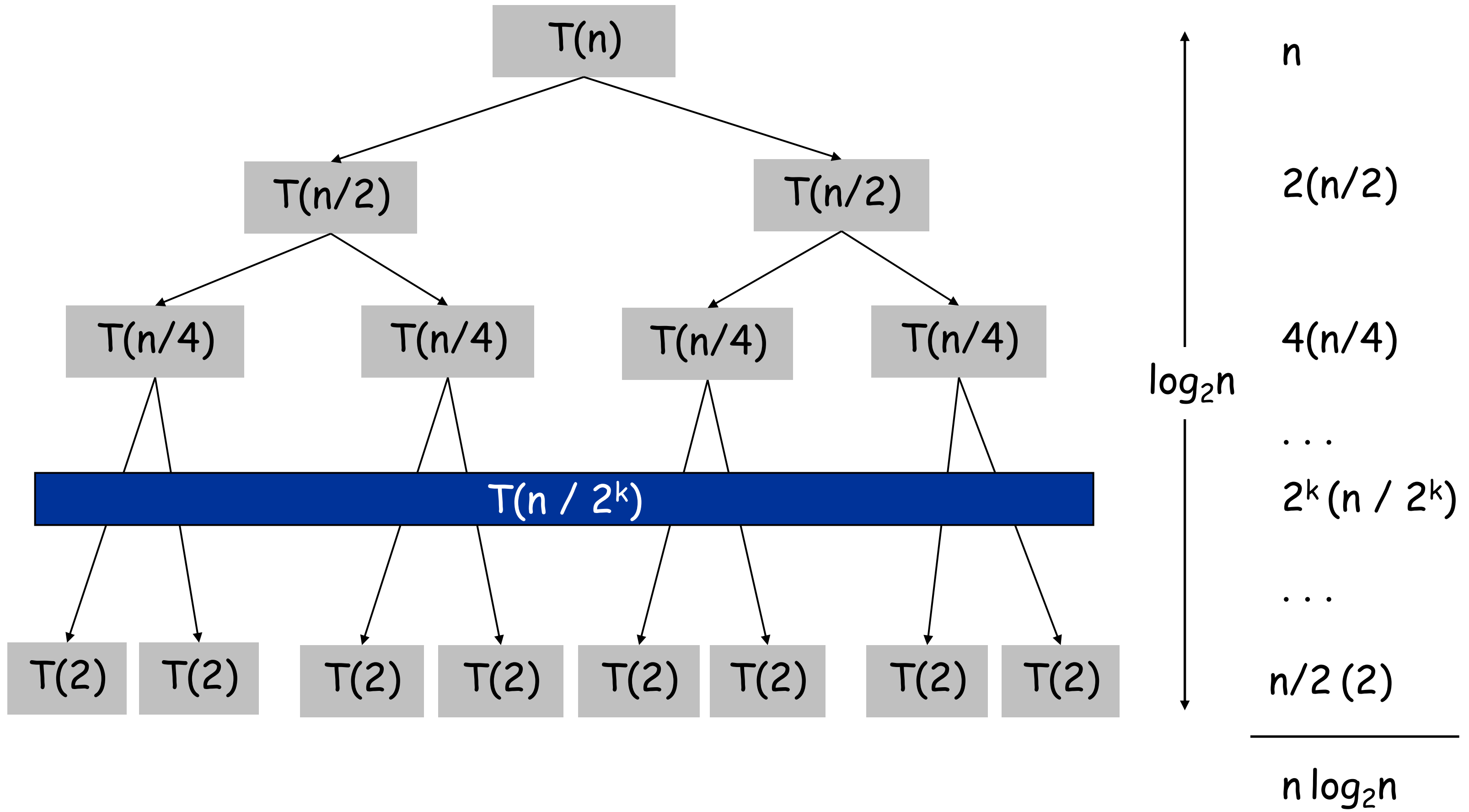


Jon von Neumann (1945)

A	L	G	O	R	I	T	H	M	S		
A	L	G	O	R	I	T	H	M	S	divide	$O(1)$
A	G	L	O	R	H	I	M	S	T	sort	$2T(n/2)$
A	G	H	I	L	M	O	R	S	T	merge	$O(n)$

Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



5.3 Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j inverted if $i < j$, but $a_i > a_j$.

Want: Count number of inversions

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Inversions

3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs i and j.

Applications

Applications.

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's Tau distance).

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: $O(1)$.

1	5	4	8	10	2

6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Conquer: 2T(n / 2)

5 blue-blue inversions

8 green-green inversions

5-4, 5-2, 4-2, 8-2, 10-2

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

5 blue-blue inversions

8 green-green inversions

Conquer: 2T(n / 2)

9 blue-green inversions
5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

Q: What happens if each half is sorted?

A: Can count blue-green inversions in O(n)

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

For j = 1 to |R|,

- Find first element L_{pos} larger than R_j; all further elements in L are larger than R_j => count(j) = $|L|$- pos.

Then add all of the count(j)'s

Notice sorting inside blue and green arrays does not change the number of blue-green inversions

Counting Inversions: Implementation

```
Count_Inversions(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (r_A) ← Count_Inversions(A)
   (A) ← Sort(A)
   (r_B) ← Count_Inversions(B)
   (B) ← Sort(B)
    r  ← Count_Inversions_Between(A, B)

   return r_A + r_B + r
}
```

$$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n \log n) \Rightarrow \mathrm{T}(n) =$$

Counting Inversions: Implementation

```
Count_Inversions(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (r_A) ← Count_Inversions(A)
   (A) ← Sort(A)
   (r_B) ← Count_Inversions(B)
   (B) ← Sort(B)
    r  ← Count_Inversions_Between(A, B)

   return r_A + r_B + r
}
```

$$T(n) \le T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + O(n \log n) \Rightarrow \mathrm{T}(n) = O(n \log^2 n)$$

Counting Inversions: Combine Revised

We want faster, O(n log n)... Cannot spend O(n log n) to sort in each recursion of **Count_Inversions**

Idea: Make **Count_Inversions** return sorted list

```
Count_Inversions(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (r_A) ← Count_Inversions(A)
   (A) ← Sort(A)
   (r_B) ← Count_Inversions(B)
   (B) ← Sort(B)
    r  ← Count_Inversions_Between(A, B)

   return r_A + r_B + r
}
```

Need to combine A, B into sorted list

Counting Inversions: Implementation

Post-condition. [Sort-and-Count] L is sorted.

```
Sort-and-Count(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (rA, A) ← Sort-and-Count(A)
   (rB, B) ← Sort-and-Count(B)
   r ← Count_Inversions_Between(A, B)
   L ← Merge(A,B)

   return r = rA + rB + r and the sorted list L
}
```

$$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) \quad \Rightarrow \quad \mathrm{T}(n) = O(n \log n)$$

Exercise

Exercise 1: You have the sequence of predicted prices p(1), p(2), ... p(n) of a single stock. Give an algorithm that tells when is the best time to buy and sell this stock (if there is no profitable buying/selling, then you don't buy).

The algorithm should be O(n log n)

From now on O(n log n) is not necessarily about sorting

Ex: p(1)=9, p(2)=1, p(3)=5   => buy on 2, sell on 3, profit of 5-1=4

Solution: [See also the solution in page 245 of the Kleinberg-Tardos book]

- Break list of prices into L = (p(1),....p(n/2)), R = (p(n/2) + 1, ..., p(n))
- Recursively find best buying/selling inside L, and inside R
- Find best way of buying inside L and selling inside R == buy in minimum price in L, sell in maximum price in R => O(n)

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Games, graphics, computer vision, geographic information systems, molecular modeling, air traffic control.

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

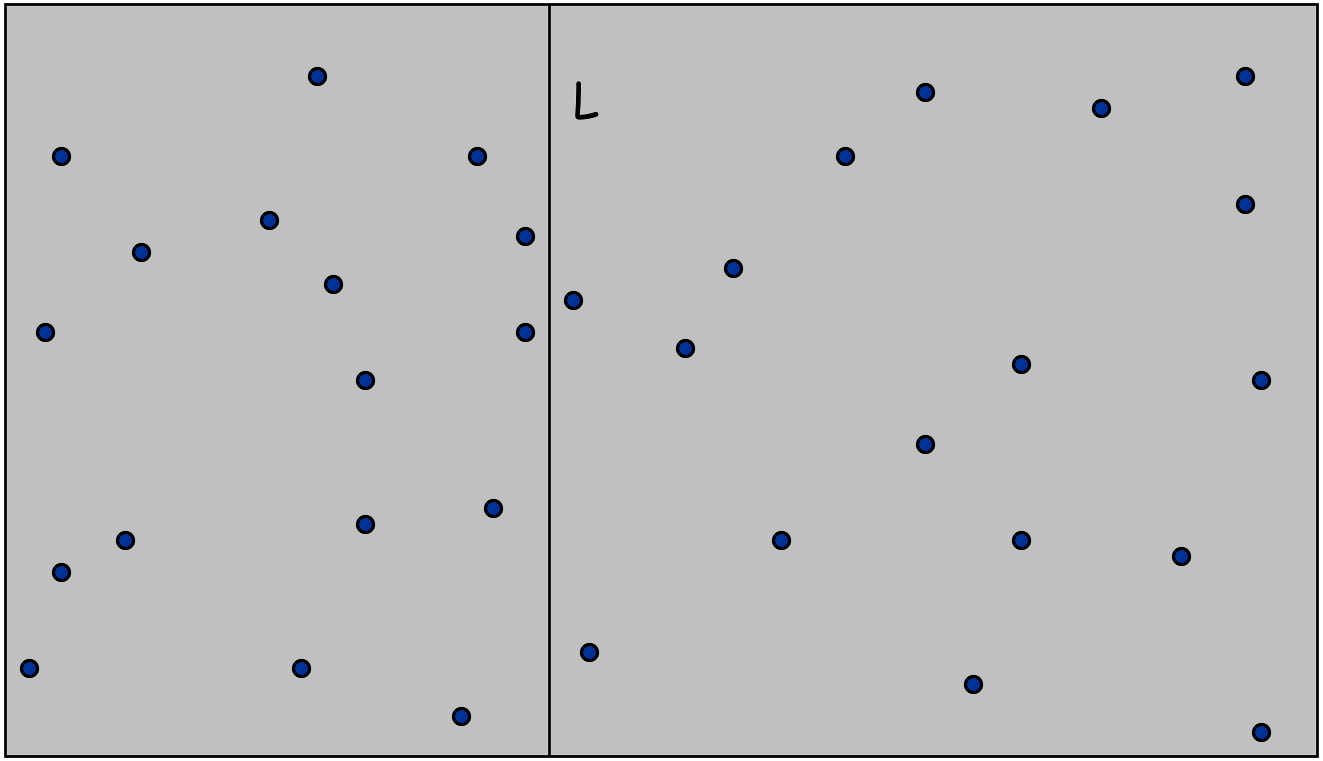
1-D version. $O(n \log n)$ easy if points are on a line.

Assumption to simplify presentation: No two points have same x coordinate.

Closest Pair of Points

Algorithm.

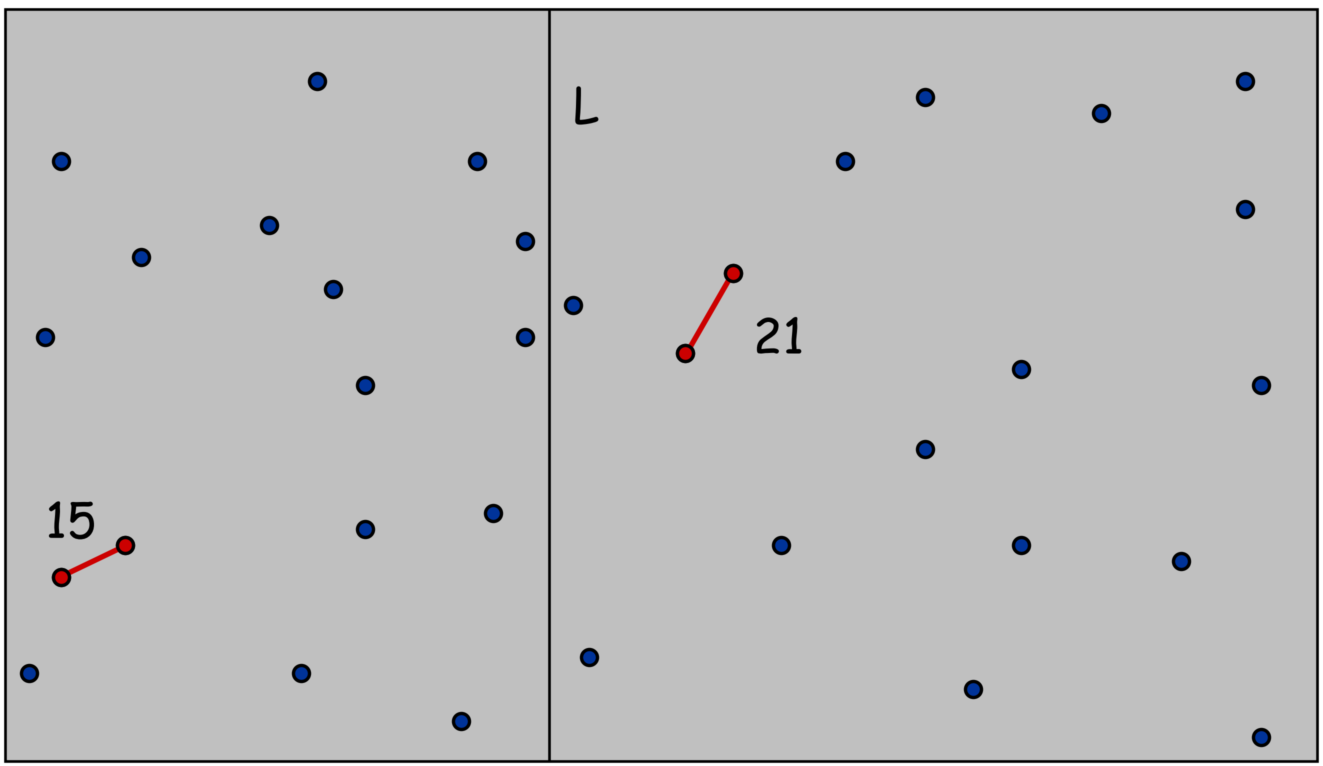
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.



Closest Pair of Points

Algorithm.

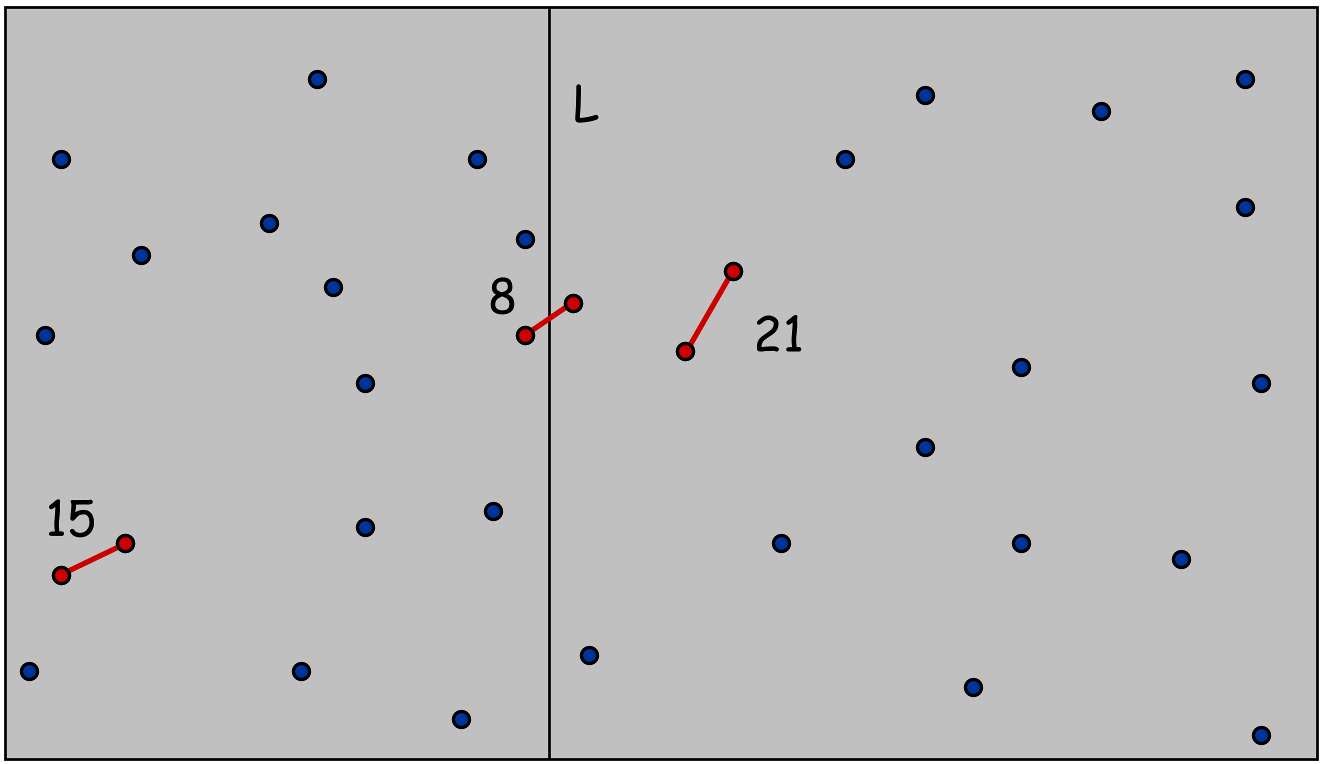
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

Algorithm.

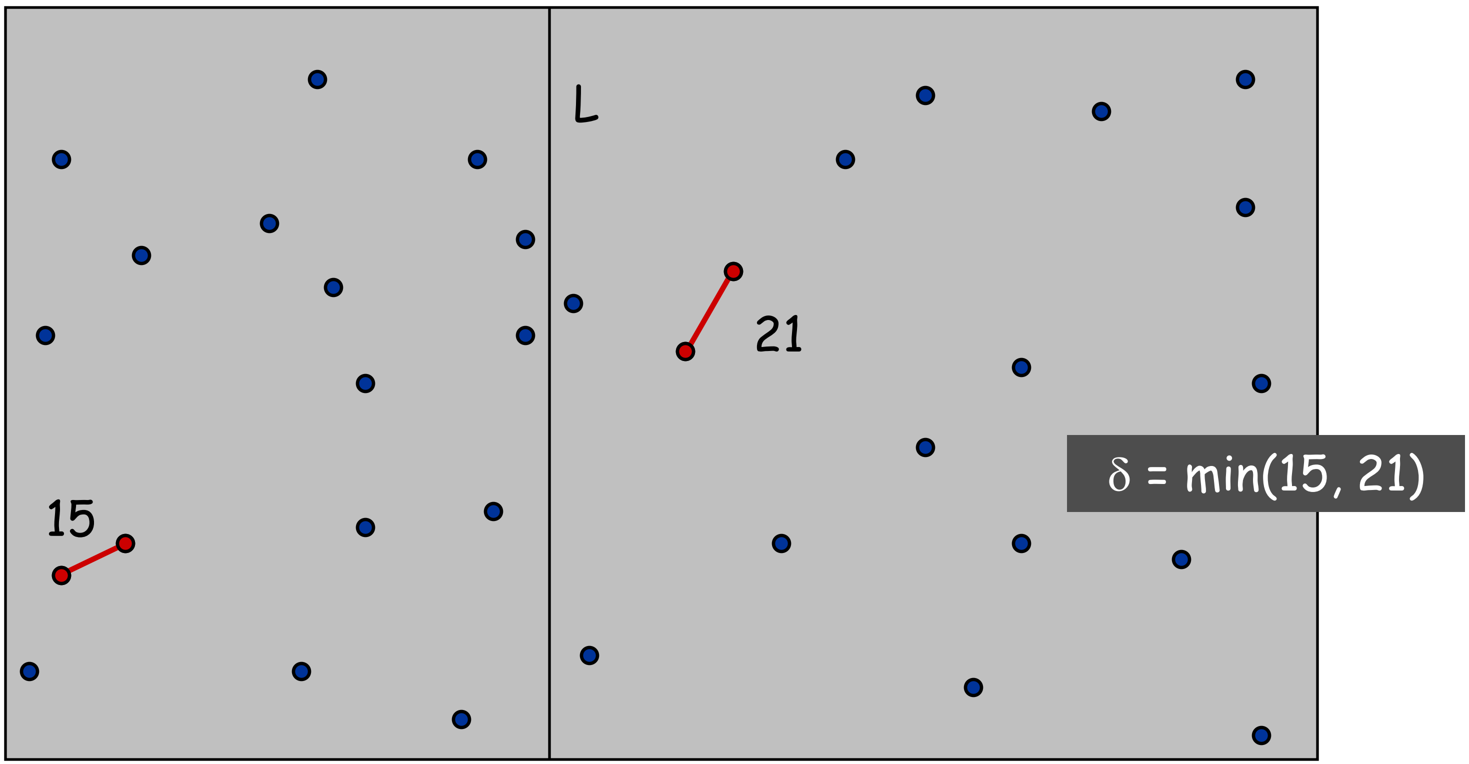
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

Let δ be the smallest closest distance found so far

Idea 1: Only need to consider pairs with one point in each side at distance less than δ.

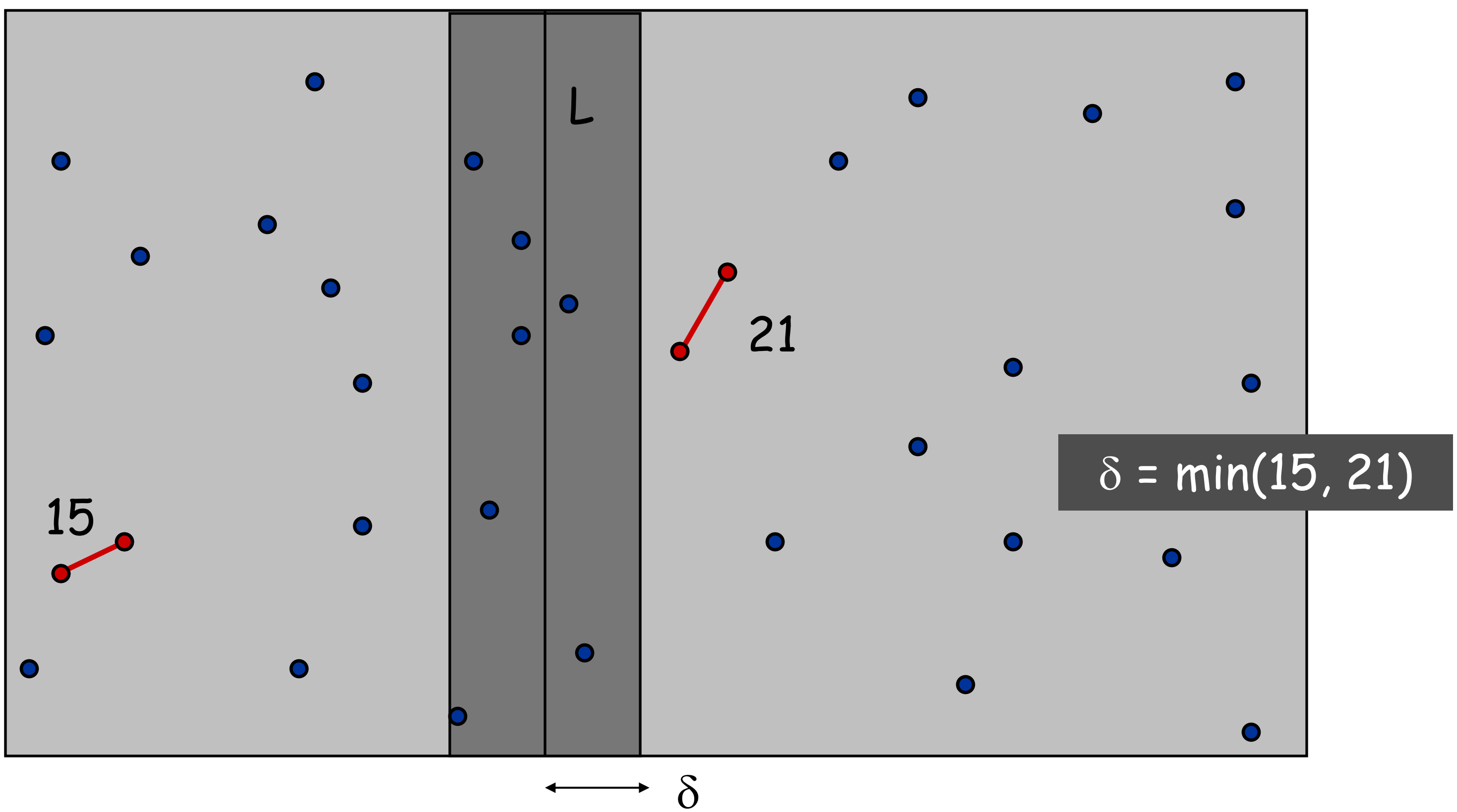


Closest Pair of Points

Let δ be the smallest closest distance found so far

Idea 1: Only need to consider pairs with one point in each side at distance less than δ.

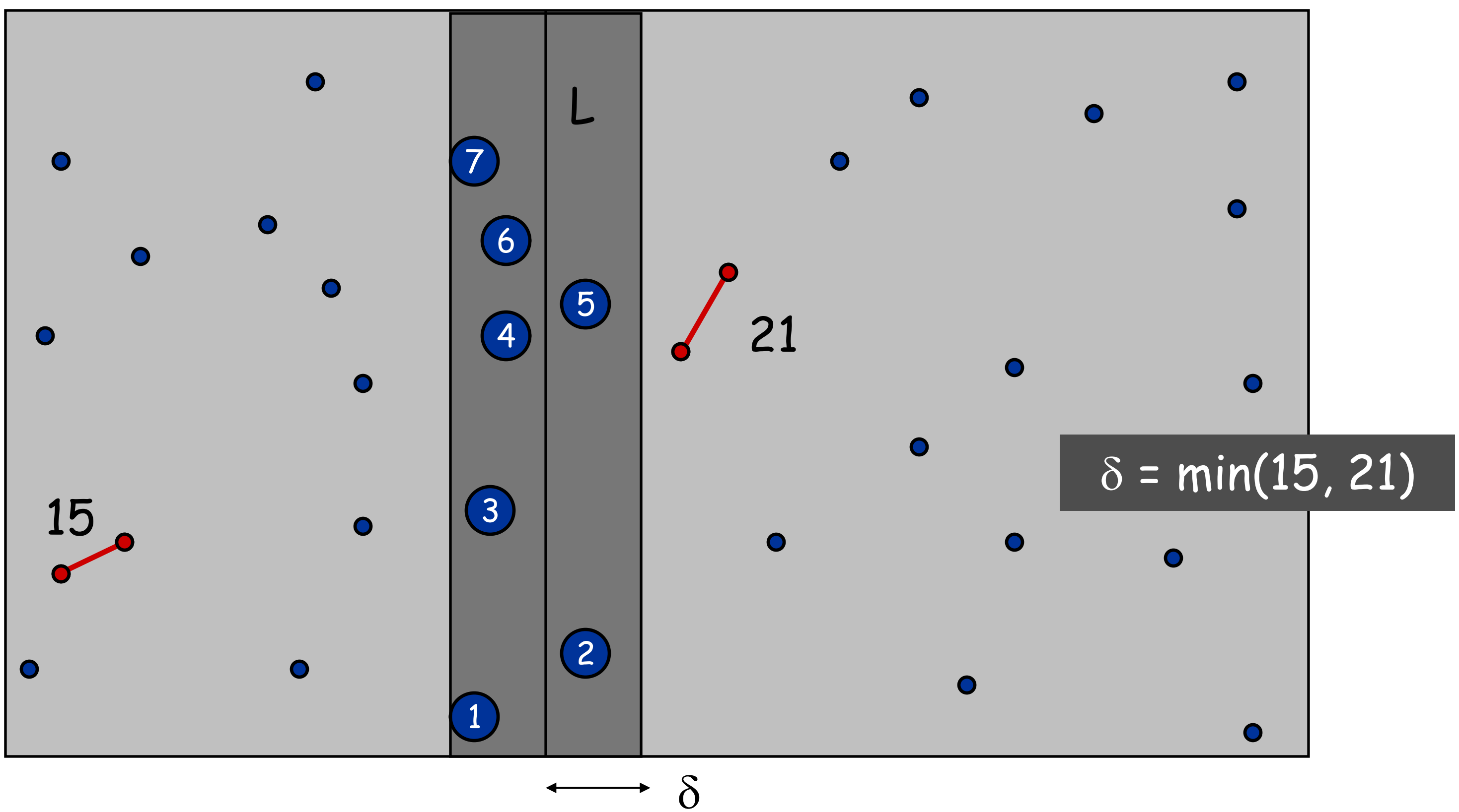
So only need to consider points within δ of line L



Closest Pair of Points

Find closest pair with one point in each side:

- Sort points in 2δ-strip by their y coordinate.

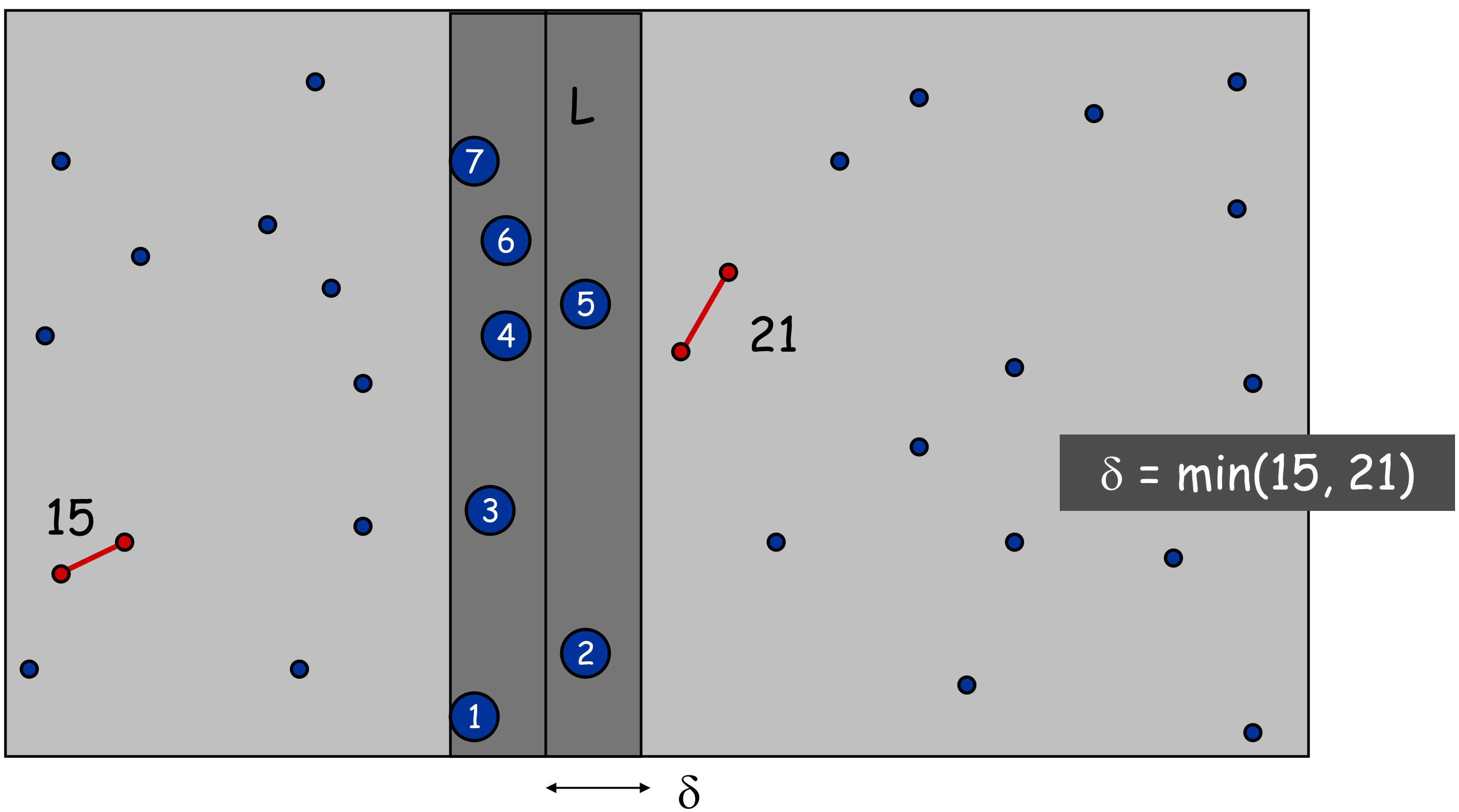


Closest Pair of Points

Find closest pair with one point in each side:

- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 12 positions in sorted list!

[So only need to compute distace between 1 and 2, 3, ..., 12
2 and 3, 4, ..., 13 ....]



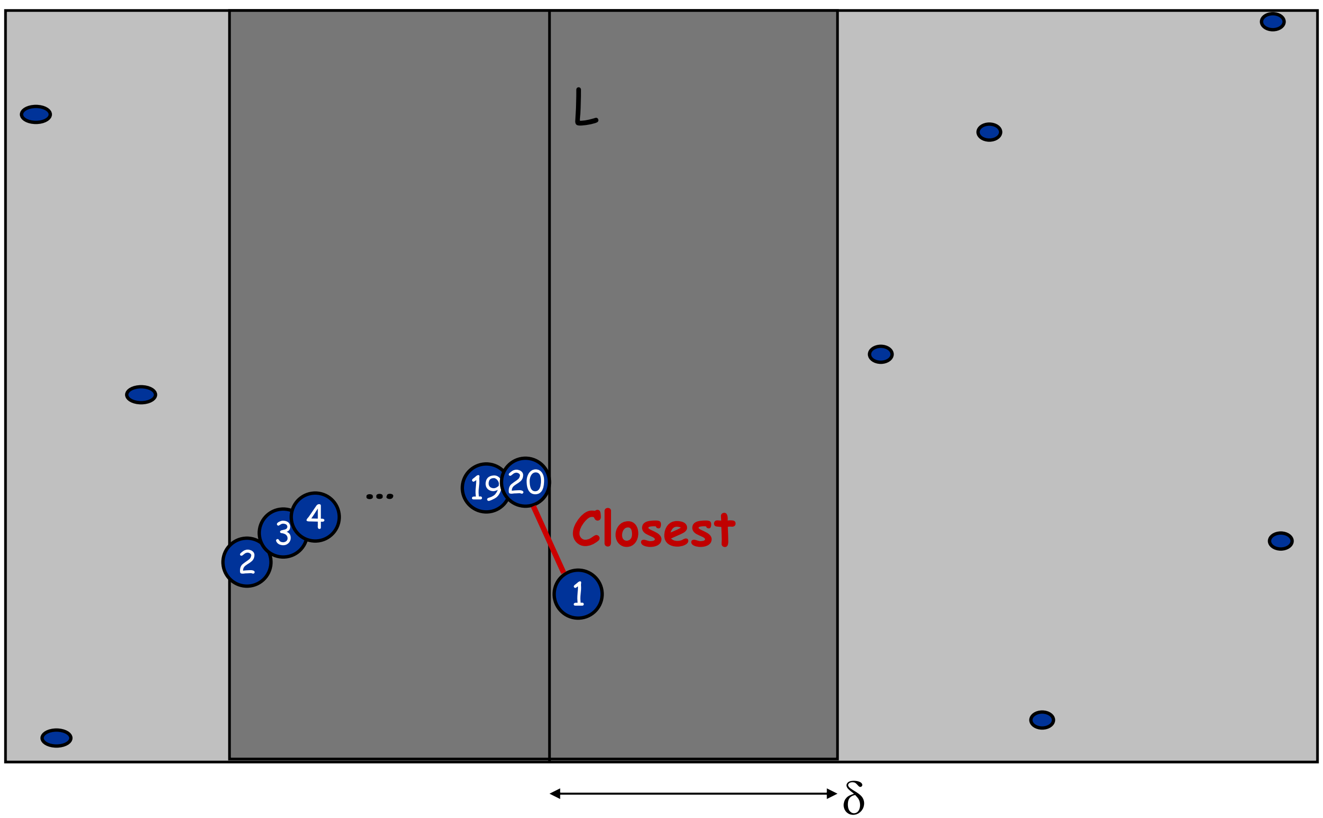
Closest Pair of Points

Find closest pair with one point in each side:

- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 12 positions in sorted list!

Q: Why is this enough? Can the following happen?

A: No! Points on each side are at distance at least δ from each other

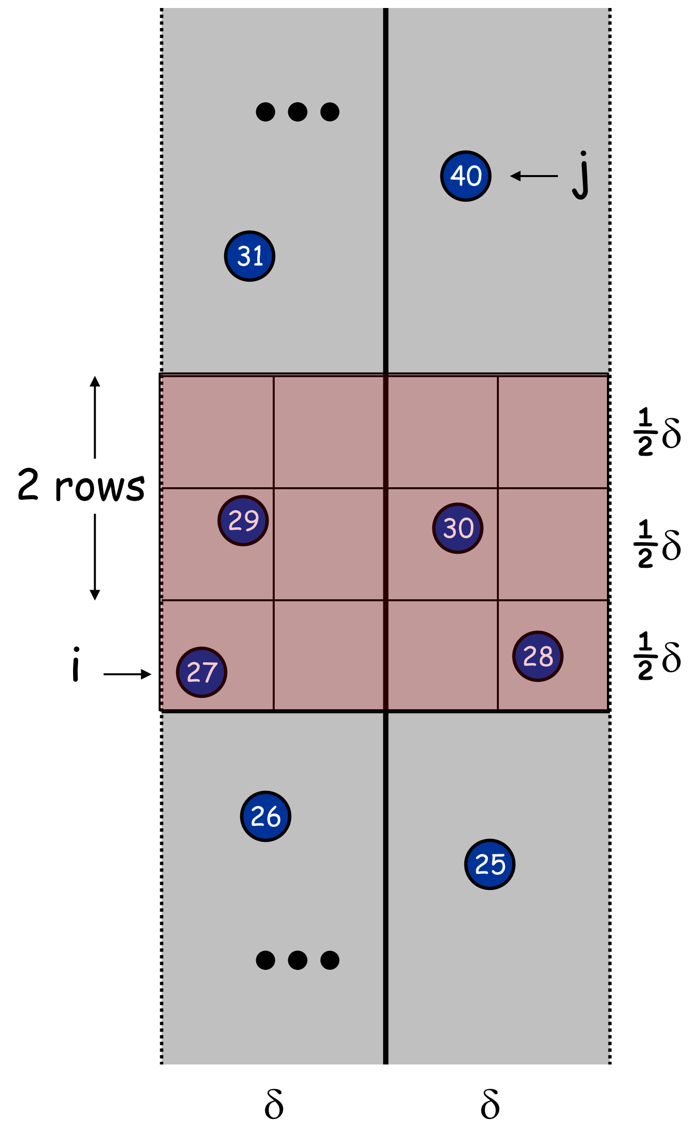


Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| > 12$, then the distance between s_i and s_j is at least δ.

Proof.
- Partition the strip into $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ boxes
- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Consider box of i-th point...
- ...box of point j=i+13 is at least 2 rows above
- => already have distance $\geq 2(\frac{1}{2}\delta)$ from i to j



Closest Pair Algorithm

Sort all points according to x-coordinate. $O(n \log n)$

```
Closest-Pair(p₁, …, pₙ) {

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)              2T(n / 2)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L      O(n)

   Sort remaining points by y-coordinate.      O(n log n)

   Scan points in y-order and compare distance between
   each point and next (in this order) 12 points.              O(n)
   If any of these distances is less than δ, update δ.

   return δ.
}
```

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Like counting inversions, don't sort points in strip from scratch each time.

- Each recursive call returns the lists of all points sorted by y coordinate
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

Closest Pair Algorithm: O(n log n)

[Shamos, Hoey '75]

Sort all points according to x-coordinate. O(n log n)

```
SortbyY_and_Closest-Pair(p1, …, pn) {

   (A,δ1)= SortbyY_and_Closest-Pair(left half)
   (B,δ2)= SortbyY_and_Closest-Pair(right half)
    δ  = min(δ1, δ2)

   S ← Merge(A,B) by y-coordinate.

   Let S' be the list obtained from S by deleting all
   points further than δ from separation line L

   Scan points of S' in y-order and compare distance
   between each point and next 12 neighbors. If any of
   these distances is less than δ, update δ.

   return δ and S.
}
```

2T(n / 2)

O(n)

O(n)

O(n)

Closest Pair of Points

Fastest algorithms

- O(n log log n) [Fortune, Hopcroft '79]
- Expected time O(n) [Khuller, Matias '95]

5.5 Integer Multiplication

Integer Arithmetic

How many bit operations we need to sum, multiply?

Add. Given two n-bit integers a and b, compute a + b.

- $O(n)$ bit operations.

Multiply. Given two n-bit integers a and b, compute a × b.

- Brute force solution: $\Theta(n^2)$ bit operations.

```
  1 1 1 1 1 1 0 1
    1 1 0 1 0 1 0 1
+   0 1 1 1 1 1 0 1
-------------------
  1 0 1 0 1 0 0 1 0
```

Add

```
                1 1 0 1 0 1 0 1 0
                * 0 1 1 1 1 1 0 1
                -----------------
                1 1 0 1 0 1 0 1 0
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
---------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1 0
```

Multiply

Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:

- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) = 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)
$$

↑

assumes n is a power of 2

Karatsuba Multiplication

To multiply two n-digit integers:

- Add two ½n digit integers.
- Multiply three ½n-digit integers.
- Add, subtract, and shift ½n-digit integers to obtain result.

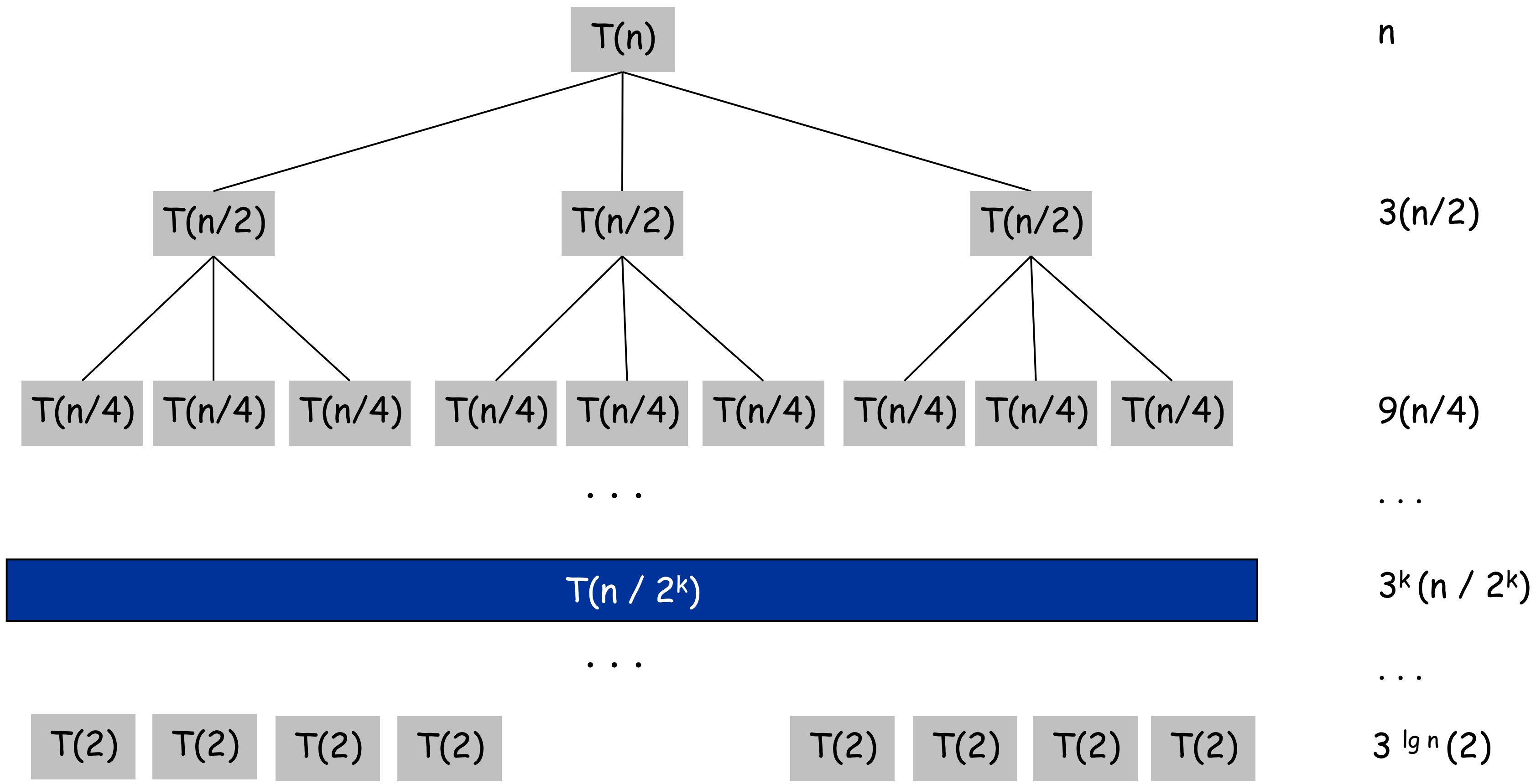
$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
&= 2^{n} \cdot \underbrace{x_1 y_1}_{A} + 2^{n/2} \cdot \left(\underbrace{(x_1 + x_0)(y_1 + y_0)}_{B} - \underbrace{x_1 y_1}_{A} - \underbrace{x_0 y_0}_{C}\right) + \underbrace{x_0 y_0}_{C}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\tfrac{3}{2}\right)^k = \frac{\left(\tfrac{3}{2}\right)^{1+\log_2 n} - 1}{\tfrac{3}{2} - 1} = 3n^{\log_2 3} - 2$$



Matrix Multiplication

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute C = AB.

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Brute force. $\Theta(n^3)$ arithmetic operations.

Fundamental question. Can we improve upon brute force?

Matrix Multiplication: Warmup

Divide-and-conquer.

- Divide: partition A and B into ½n-by-½n blocks.
- Conquer: multiply 8 ½n-by-½n recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$\mathrm{T}(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow \mathrm{T}(n) = \Theta(n^3)$$

Matrix Multiplication: Key Idea

Key idea. multiply 2-by-2 block matrices with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

- 7 multiplications.
- 18 = 10 + 8 additions (or subtractions).

Fast Matrix Multiplication

Fast matrix multiplication. (Strassen, 1969)

- Divide: partition A and B into ½n-by-½n blocks.
- Compute: 14 ½n-by-½n matrices via 10 matrix additions.
- Conquer: multiply 7 ½n-by-½n matrices recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- T(n) = # arithmetic operations.

$$\mathrm{T}(n) = \underbrace{7T\big(n/2\big)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \;\Rightarrow\; \mathrm{T}(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Fast Matrix Multiplication in Practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.

Common misperception: "Strassen is only a theoretical curiosity."

- Advanced Computation Group at Apple Computer reports 8x speedup on G4 Velocity Engine when n ~ 2,500.
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize" Ax=b, determinant, eigenvalues, and other matrix ops.

Fast Matrix Multiplication in Theory

Q. Multiply two 2-by-2 matrices with only 7 scalar multiplications?
A. Yes! [Strassen, 1969]

Q. Multiply two 2-by-2 matrices with only 6 scalar multiplications?
A. Impossible. [Hopcroft and Kerr, 1971]

Q. Two 3-by-3 matrices with only 21 scalar multiplications?
A. Also impossible.

Q. Two 70-by-70 matrices with only 143,640 scalar multiplications?
A. Yes! [Pan, 1980]

Decimal wars.
- December, 1979: $O(n^{2.521813})$.
- January, 1980: $O(n^{2.521801})$.

Fast Matrix Multiplication in Theory

Best known. $\cancel{O(n^{2.376})}$ [Coppersmith-Winograd, 1987.]

$O(n^{2.3728639})$ [Williams '13, Le Gall '14]

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. Theoretical improvements to Strassen are progressively less practical.

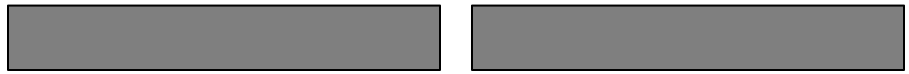
Exercise

Max interval sum: Given a list of positive and negative values, compute the interval with largest sum

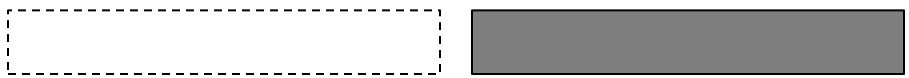
Exercises of Kleinberg-Tardos (Chapter 5): 2, 3

Another type of divide-and-conquer: binary search

In all examples so far we have recursed on both halves of the instance



But another important class of div-and-conquer algorithms are like binary search: only recurse on one half



Typically these have running time O(log n)

Another type of divide-and-conquer: binary search

Exercise: Consider a sequence of distinct numbers a_1, a_2, …, a_n that is unimodal, that is, it is increasing up to some point ai, and then decreasing after that. This maximum element ai is called the mode.

Give a O(log n) algorithm for finding the mode of the sequence

A: Look at the middle of the sequence and check if it the mode:

if $a_{n/2-1}$ < $a_{n/2}$ > $a_{n/2+1}$, return $a_{n/2}$

Else, compute "slope" in the middle of the sequence to see which half should recurse on:

if $a_{n/2}$ > $a_{n/2+1}$, recurse on a_1,…,$a_{n/2}$
else recurse on $a_{n/2+1}$, … , a_n

Another type of divide-and-conquer: binary search

Exercise: Consider a strictly increasing sequence $a_1, a_2, \ldots, a_n$ of integers. Find if there is an element with $a_i = i$ in time O(log n).

A: Look at the middle of the sequence:

if $a_i > i$, since strictly increasing we have $a_{i+1} > i+1$, $a_{i+2} > i+2\ldots$
=> recurse on first half of sequence

if $a_i < i$, similarly we have $a_{i-1} > i-1$, $a_{i-2} > i-2\ldots.$
=> recurse on second half of sequence