1.
Basics of
Algorithm Analysis

Time Complexity of an Algorithm

Purpose

- To estimate how long a program will run
- To estimate the largest input that can reasonably be given to the program
- To compare the efficiency of different algorithms
- To choose an algorithm for an application

Time complexity is a function

Time for a sorting algorithm is different for sorting 10 numbers and sorting 1,000 numbers

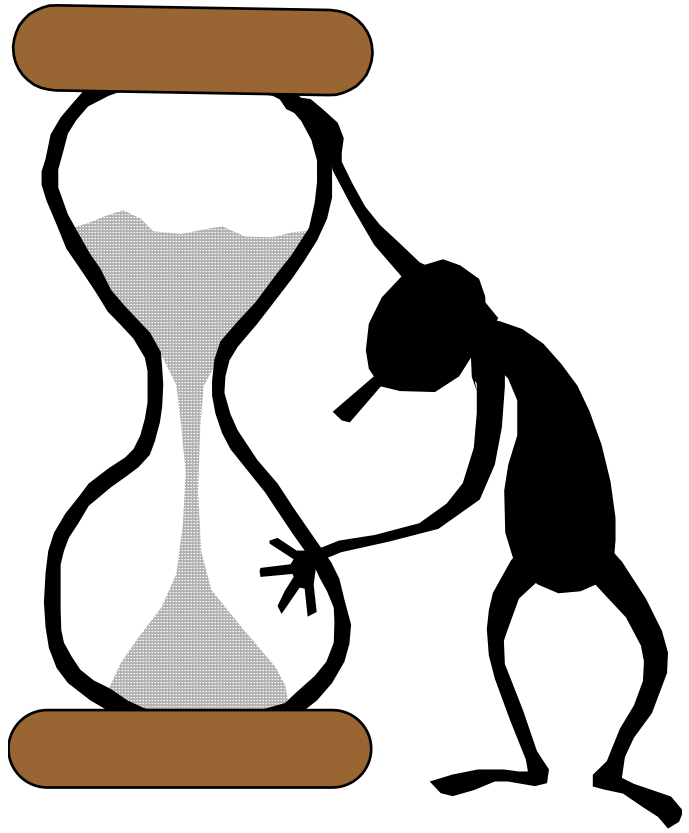
Time complexity is a function: Specifies how the running time depends on the size of the input.

Function mapping

"size" n of input

"time" $T(n)$ executed by algorithm

Definition of time?



Definition of time?

- # of seconds
- # lines of code executed
- # of simple operations performed

Definition of time?

- # of seconds Problem: machine dependent
- # lines of code executed Problem: lines of diff. complexity
- # of simple operations performed

this is what we will use

Size of input instance?

Formally: Size n is number of bits to represent instance

But we can work with anything reasonable

reasonable = within a constant factor of number of bits

Size of input instance

Ex 1:

- # of bits: 17 bits - Formal
- # of digits: 5 digits - Reasonable: #bits and #digits are always within constant factor $\approx \log_2 10 \approx 3.32$
- Value: 83920

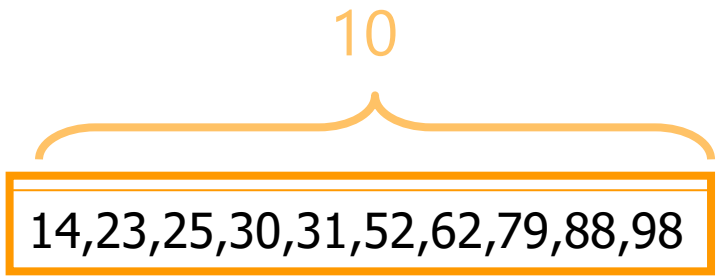
Size of input instance

Ex 1:

- # of bits: 17 bits - Formal
- # of digits: 5 digits - Reasonable
- Value: 83920 - Not reasonable: $\approx 2^{\#\text{bits}}$, much bigger

Size of input instance

Ex 2:

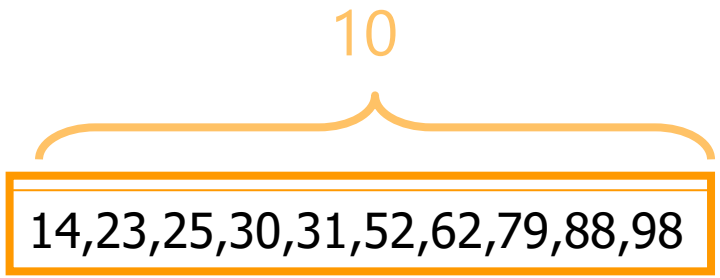


- # of elements = 10

Is this reasonable?

Size of input instance

Ex 2:



- # of elements = 10 - Reasonable if each number is, say, a 32-bit word, total number of bits is

  #bits = 32 * #elements

Time complexity is a function

Time complexity is a function: Specifies how the running time depends on the size of the input

Function mapping

# of bits n to represent input

# of basic operations $T(n)$ executed by the algorithm

Which input of size n?

Q: There are 2^n inputs of size n. Which do we consider for the time complexity T(n)?

Worst instance

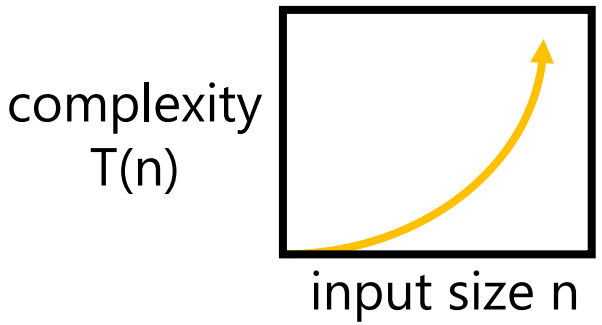
Worst-case running time. Consider the instance where the algorithm uses largest number of basic operations

- Generally captures efficiency in practice
- Pessimistic view, but hard to find better measure

Time complexity

We reach our final definition of time complexity:

T(n) = number of basic operations the algorithm takes over the worst instance of bit-size n



Example 1

```
Func Algorithm 1(A)        #A is array of bits
    x=20
    For i=1...len(A)
        x=3x
```

Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) \approx 2n + 1$

- Input A of bit-size *n* has *n* entries
- ≈ 2 simple operations per step of **for**, +1 for "x=20"

(ignoring extra operations that make up the **For**)

Example 2

```
Func Algorithm 2(A)          #A is array of bits
    x=1
    For i=1...len(A)
      x=x+1
      If x>50 then
        x=x+3
      End If
    End For
```

Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) \approx 5n - 99$

- Input A of bit-size *n* has *n* entries
- +1 for initialization "x=1"
- … ≈ 2+1 per iterations of **for** in the first 50 iterations
- … ≈ 2+1+2 per iterations of **for** in the other (n-50) iterations

(assuming $n \geq 50$)

Example 3

```
Func Algorithm 3(A)        #A is array of 32-bit numbers
    For i=1 to len(A)
        print "oi"
```

Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) \approx (n/32)$

- A of bit-size *n* has *n/32* numbers
- ≈ 1 simple operations per iteration of **for**

Point: Understand input size

Example 4

```
Func Find10(A)                    #A is array of 32-bit numbers
    For i=1 to len(A)
        If A[i]==10
            Return i
```

Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) \approx (n/32) + 1$

- Worst instance: the only "10" is in the last position
- A of bit-size *n* has *n/32* numbers
- ≈ 1 simple operations per **for**, +1 for **Return**

Point: Complexity of algo is **always** about the **worst instance**

Asymptotic Order of Growth

Asymptotic Order of Growth

Motivation: Determining the exact time complexity T(n) of a real algorithm is very hard and often does not make much sense

In particular, can we say that an algorithm with complexity

$$T(n) = 10n$$

will run slower than an algorithm with complexity

$$T(n) = 9n?$$

No; for example, maybe the opertions in the first algorithm are slightly faster then in the second
(e.g., addition vs. multiplication)

But as we will see, for large instances there is a difference between $\approx n$ and $\approx n^2$
(e.g., ≈ 1.000 vs $\approx 1.000.000$)

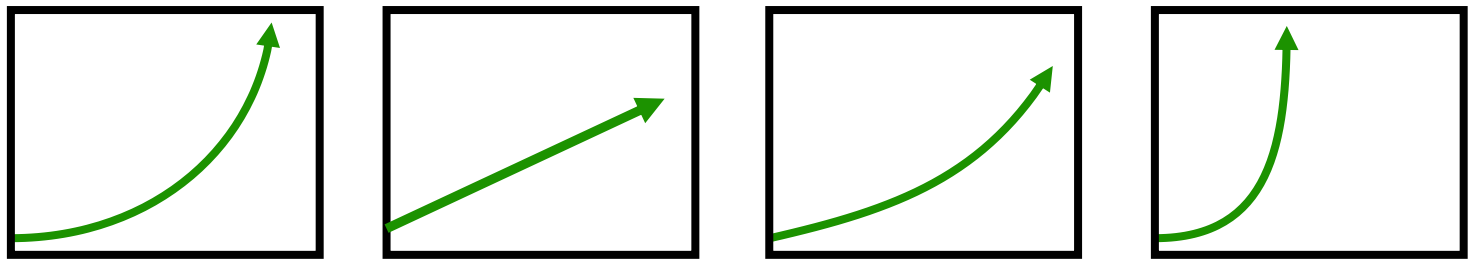
Asymptotic Order of Growth

We will focus on the asymptotic order of growth of the complexity T(n)

So T(n)=$30n^2 + 7n + 10$ will become T(n)= $\theta(n^2)$

We just want to differentiate T(n)=

$\sim n^2$ vs $\sim$n vs $\sim$n log n vs $\sim 2^n$ ....



Asymptotic Order of Growth

Actually, to compute the asymptotic order of growth of T(n) we will compute upper and lower bounds for T(n):

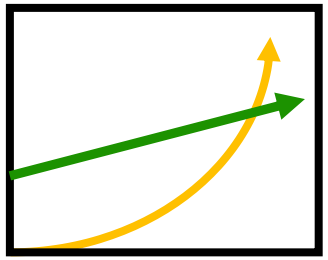
Ex: T(n) grows at most (not faster) like n^2
T(n) grows at least like n^2

T(n) grows just like n^2

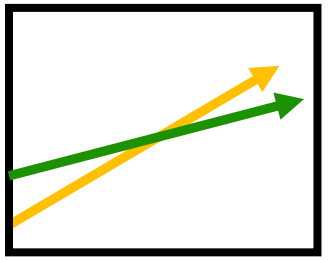
Asymptotic Order of Growth: Upper Bounds

Upper bounds

Informal: T(n) is O(f(n)) if T(n) grows with at **most** the same order of magnitude as f(n) grows



T(n) is O(f(n))



T(n) is O(f(n))

both grow at same order of magnitude

Asymptotic Order of Growth: Upper Bounds

Upper bounds

Formal: T(n) is O(f(n)) if there exist a constant $c \geq 0$ such that for all $n \geq 1$ we have

$$T(n) \leq c \cdot f(n).$$

Equivalent: T(n) is O(f(n)) if there exists $c \geq 0$ such that

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} \leq c$$

Asymptotic Order of Growth: Upper Bounds

Exercise 1: $T(n) = 32n^2 + 17n + 32$.

Say if T(n) is:

- $O(n^2)$?
- $O(n^3)$?
- $O(n)$?

Asymptotic Order of Growth: Upper Bounds

Exercise 1: $T(n) = 32n^2 + 17n + 32$.

Say if T(n) is:

- $O(n^2)$? Yes
- $O(n^3)$? Yes
- $O(n)$? No

Solution: To show that T(n) is $O(n^2)$ we can:

- Use the first definition with c = 1000
- Use limits: $\lim_{n\to\infty} \frac{T(n)}{n^2} = 32$, which is a constant

Asymptotic Order of Growth: Upper Bounds

Exercise 2:

- $T(n) = 2^{n+1}$, is it $O(2^n)$?
- $T(n) = 2^{2n}$, is it $O(2^n)$?

Asymptotic Order of Growth: Upper Bounds

Exercise 2:

- $T(n) = 2^{n+1}$, is it $O(2^n)$? Yes
- $T(n) = 2^{2n}$, is it $O(2^n)$? No

Solution (second item): $\lim_{n\to\infty} \frac{T(n)}{2^n} = \lim_{n\to\infty} 2^n = \infty$ is not constant

Solution 2 (second item): To have $2^{2n} < c.2^n$ we need $c > 2^n$. So c is not a constant

Upper Bounds Involving log/exp

Logarithms. $\log_a n$ is $O(\log_b n)$ for any constants $a, b > 0$

can avoid specifying the base

Logarithms. For every $x > 0$, $\log n$ is $O(n^x)$

log grows slower than every polynomial

Exponentials. For every $r > 1$ and every $d > 0$, n^d é $O(r^n)$

every exponential grows faster than every polynomial

Upper Bounds Involving log/exp

Exercise: is T(n) = 21*n*log n

- $O(n^2)$?
- $O(n^{1.1})$?
- $O(n)$?

Upper Bounds Involving log/exp

Exercise: is T(n) = 21*n*log n

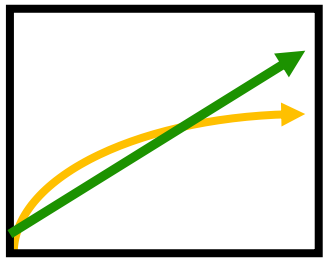
- O(n^2) ? Yes
- O($n^{1.1}$) ? Yes
- O(n) ? No

Solution (first item): Comparing 21*n*log n vs. n^2 is the same as comparing 21*log n vs. n, and we know log n grows slower than n

Solution 2 (first item): $\lim_{n\to\infty} \frac{T(n)}{n^2} = \lim_{n\to\infty} \frac{21\log n}{n}$, which is at most a constant since log n grows slower than n

Lower Bounds

Informal: T(n) is $\Omega(f(n))$ if T(n) grows with at **least** the same order of magnitude as f(n) grows



Formal: T(n) is $\Omega(f(n))$ if there exist constants $c > 0$ such that for all n we have $T(n) \geq c \cdot f(n)$.

Equivalent: T(n) is $\Omega(f(n))$ if there exist constant $c>0$

$$\lim_{n\to\infty} \frac{T(n)}{f(n)} \geq c$$

Tight Bounds

Tight bounds. T(n) is $\Theta(f(n))$ if T(n) is both $O(f(n))$ and $\Omega(f(n))$

T(n) grows at most as fast as f(n) — T(n) is $O(f(n))$

T(n) grows at least as fast as f(n) — T(n) is $\Omega(f(n))$

T(n) grows just like f(n) — T(n) is $\Theta(f(n))$

Lower and Tight Bounds

Exercise: $T(n) = 32n^2 + 17n + 32$

Is T(n):

- $\Omega(n)$?
- $\Omega(n^2)$?
- $\Theta(n^2)$?

- $\Omega(n^3)$?
- $\Theta(n)$?
- $\Theta(n^3)$?

Lower and Tight Bounds

Exercise: $T(n) = 32n^2 + 17n + 32$
Is T(n):

- $\Omega(n)$? Yes
- $\Omega(n^2)$? Yes
- $\Theta(n^2)$? Yes

- $\Omega(n^3)$? No
- $\Theta(n)$? No
- $\Theta(n^3)$? No

Solution (second item): $\lim_{n\to\infty} \frac{T(n)}{n^2} = 32$ is constant > 0

Solution 2 (second item): To show T(n) is $\Omega(n^2)$ use c = 1

Back to algorithms

```
Func Algorithm 1(A)        #A is array of bits
    x=20
    For i=1...len(A)
        x=3x
```

Q: What is asymptotic time complexity $T(n)$ of this algorithm?

A: $T(n) = \Theta(n)$

- Just notice/remember $T(n) \approx 2n + 1$

Back to algorithms

```
Func Algorithm 1(A)        #A is array of bits
    x=20
    For i=1...len(A)
        x=3x
```

Q: What is asymptotic time complexity $T(n)$ of this algorithm?

A: $T(n) = \Theta(n)$

- Input A of bit-size *n* has *n* entries, so *n* iterations of **for**
- The algorithm makes at most 10n operations ⇒ $T(n)$ is $O(n)$
- The algorithm makes at least n operations ⇒ $T(n)$ is $\Omega(n)$
- So $T(n) = \Theta(n)$

Back to algorithms

```
Func Find10(A)                    #A is array of 32-bit numbers
    For i=1 to len(A)
        If A[i]==10
            Return i
```

Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) = \Theta(n)$

- Remember need to look at worst instance to get T(n)
- Notice/remember that $T(n) \approx (n/32) + 1$

Back to algorithms

```
Func Find10(A)                    #A is array of 32-bit numbers
    For i=1 to len(A)
        If A[i]==10
            Return i
```

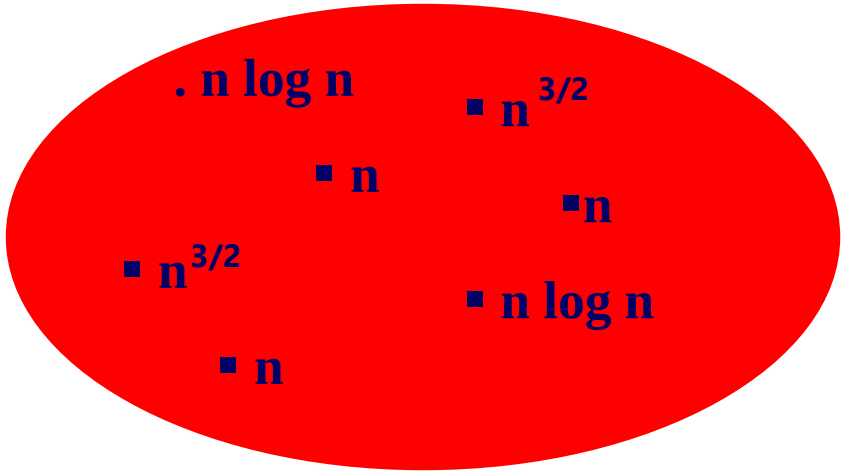
Q: What is the time complexity $T(n)$ of this algorithm?

A: $T(n) = \Theta(n)$

- Remember need to look at worst instance to get T(n)
- Worst instance: the only "10" is in the last position
- A of bit-size *n* has *n/32* numbers (using formal definition)
- The algorithm makes at most 5n operations ⇒ $T(n)$ is $O(n)$
- The algorithm makes at least n/32 operations (remember worst instance) ⇒ $T(n)$ is $\Omega(n)$
- So $T(n) = \Theta(n)$

Back to algorithms

inputs of size n
for algorithm A
(cartoon)

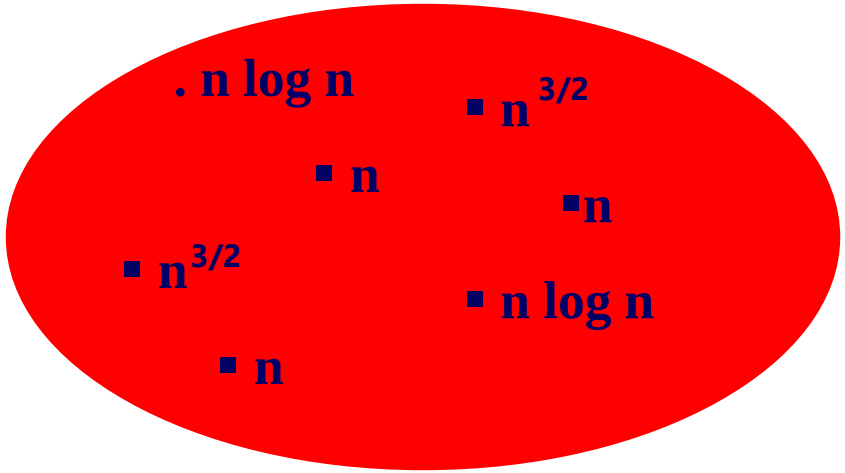


Can we say that the time complexity of A is?

- O(n^2) ?
- Ω(n^2) ?
- Ω (n) ?
- O (n) ?
- Ω ($n^{3/2}$) ?

Back to algorithms

inputs of size n for algorithm A (cartoon)



Can we say that the time complexity of A is?

- O(n^2) ? Yes, beccause largest complexity of algorithm is at most n^2
- Ω(n^2) ? No, there is no input where the complexity of the algorithm has order n^2
- Ω (n) ? Yes
- O (n) ? No, there are inputs where complexity has larger order
- Ω ($n^{3/2}$) ? Yes

What does asymptotic analysis give us?

Does not tell that exact constants in the time-complexity of an algo

Does give a good basis of comparison between algorithms

- Even if optimize implementation of $\Theta(n^2)$ algorithm and make it 10x faster, it is probably much slower than a "bad" implementation of a $\Theta(n)$ algorithm (for large instances)

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

A Survey of Common Running Times

Linear Time: O(n)

Linear time. Running time is at most a constant factor times the size of the input.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max ← a_1
for i = 2 to n {
   if (a_i > max)
      max ← a_i
}
```

Remark. For all instances the algorithm executes a linear number of operations

Linear Time: O(n)

Linear time. Running time is at most a constant factor times the size of the input.

Finding an item x in a list. Test if x is in the list $a_1, \ldots, a_n$

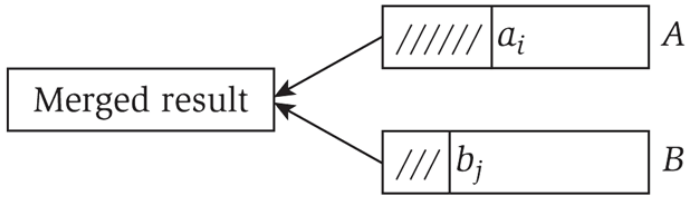
```
Exist ← false
for i = 1 to n {
   if (a_i == x)
      Exist ← true
      break

}
```

Remark. For some instances the algorithm is sublinear (e.g. x in the first position)

Linear Time: O(n)

Merge. Combine two sorted lists $A = a_1, a_2, \ldots, a_k$ with $B = b_1, b_2, \ldots, b_k$ (increasing order) into sorted whole.




```
i = 1, j = 1
while (I <= |A| and j <= |B|) {
   if (a_i ≤ b_j) append a_i to output list and increment i
   else           append b_j to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size k takes O(n) time (n=total size=2k).

Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

Sorting. Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps x_1, ..., x_n on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane (x_1, y_1), …, (x_n, y_n), find the distance of the closest pair.

$O(n^2)$ solution. Try all pairs of points.

```
min ← sqrt((x1 - x2)^2 + (y1 - y2)^2)
for i = 1 to n-1 {
   for j = i+1 to n {
      d ← sqrt((xi - xj)^2 + (yi - yj)^2)
      if (d < min)
         min ← d
   }
}
```

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Let $S_1, \ldots, S_n$ be subsets of {1, 2, …, n}. Is there a disjoint pair of sets?

Set Representation. Assume that each set is represented as an incidence vector.

n=8 and S={2,3,6}, S is represented by (0,1,1,0,0,1,0,0)

n=8 and S={1,4}, S is represented by (1,0,0,1,0,0,0,0)

Algorithm:

```
For i=1...n-1
        For j=i+1...n
            If Disjoint(i, j)
                Return 'There are disjoint sets'
            End If
        End For
End For
Return 'There are no disjoint sets'
```

Disjoint(i, j):

```
k←1
While k<=n
    If S_i(k)=S_j(k)=1 Return False
    k++
End While
Return True
```

Cubic Time: $O(n^3)$

1. A complexidade de tempo do algoritmo é $O(n^3)$?

2. A complexidade de tempo do algoritmo é $\Omega(n^3)$?

Cubic Time: $O(n^3)$

1. A complexidade de tempo do algoritmo é $O(n^3)$? SIM

2. A complexidade de tempo do algoritmo é $\Omega(n^3)$? SIM

"Bad" instance: all sets are equal to {n} => algoritm makes $\Omega(n^3)$ basic operations

Exponential Time

Independent set. Given a graph, find the largest independent set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

```
S* ← ϕ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```

Polynomial Time

Polynomial time. Running time is $O(n^d)$ for some constant d independent of the input size n.

Ex: $T(n) = 32n^2$ and $T(n) = n \log n$ are polynomial time

We consider an algorithm efficient if time-complexity is polynomial

efficient

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
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Complexity of Algorithm vs Complexity of Problem

There are many different algorithms for solving the same problem

Showing that an algorithm is $\Omega(n^3)$ does not mean that we cannot find another algorithm that solves this problem faster, say in $O(n^2)$

Exercises

Exercicio 1. Analise a complexidade de pior caso do algoritmo abaixo. Ou seja, encontre uma funcao $f(n)$ tal que $T(n) = \Theta\big(f(n)\big)$. Justifique.

```
t ← 0                                   } cst
Cont ← 1                                }
Para  i=1 até n                         } cst*n
        Cont ← cont+1                   }
Fim Para                                }
Enquanto cont ≥ 1                       } log(n) iterations * [cst*n per iteration]
        Cont ← cont/2                   } = cst*n*log(n)
        Para j = 1 a n                  } cst*n
                t++                     }
        Fim Para                        }
Fim Enquanto                            }
```

Solucao: O algoritmo é $\Theta(n \log n)$

Exercises

Exercício 2. Considere um algoritmo que recebe um número real x e o vetor $(a_0,a_1,\ldots,a_{n-1})$ como entrada e devolve

$$a_0 + a_1x + \ldots + a_{n-1}x^{n-1}$$

a) Desenvolva um algoritmo para resolver este problema que execute em tempo **quadrático**. Faça a análise do algoritmo

b) Desenvolva um algoritmo para resolver este problema que execute em tempo **linear**. Faça a análise do algoritmo

Exercises

Exercícios Kleinberg & Tardos, cap 2 da lista de exercícios

Exercises

a)

```
sum = 0
Para i= 0 até n-1 faça
        aux ← a_i
        Para j:=1 até i
                aux ← x . aux
        Fim Para
        sum ← sum + aux
 Fim Para
 Devolva sum
```

- Análise

Número de operações elementares é igual a

$$1+2+3+ \ldots + n-1 = n(n-1)/2 = O(n^2)$$

Exercises

b)

sum = a_0
pot = 1
Para i= 1 até n-1 **faça**
    pot ← x.pot
    sum ← sum + a_i.pot
Fim Para
Devolva sum

- Análise

A cada loop são realizadas O(1) operações elementares. Logo, o tempo é linear

2.5 A First Analysis of a Recursive Algorithm: Binary Search

Binary Search

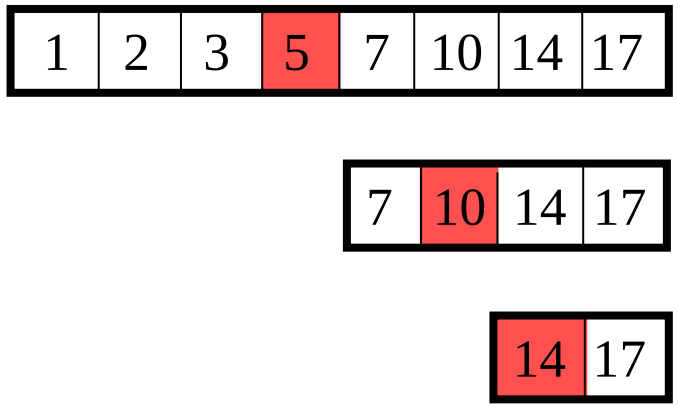
Problem: Given a sorted list of numbers (increasing order) a1,…an, decide if number x is in the list

```
Function bin_search(i,j, x)
   if i = j
      if a_i = x return TRUE
      else return FALSE
   end if

   mid = floor((i+j)/2)
   if x = a_mid
      return TRUE
   else if x < a_mid
      return bin_search(i, mid-1, x)
   else if x > a_mid
      return bin_search(mid+1, j, x)
   end if
```

```
Function bin_search_main(x)
   bin_search(1,n, x)
```

Ex: x=14



Binary Search Analysis

Binary search recurrence: $T(n) \leq c + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$

we will always ignore floor/ceiling

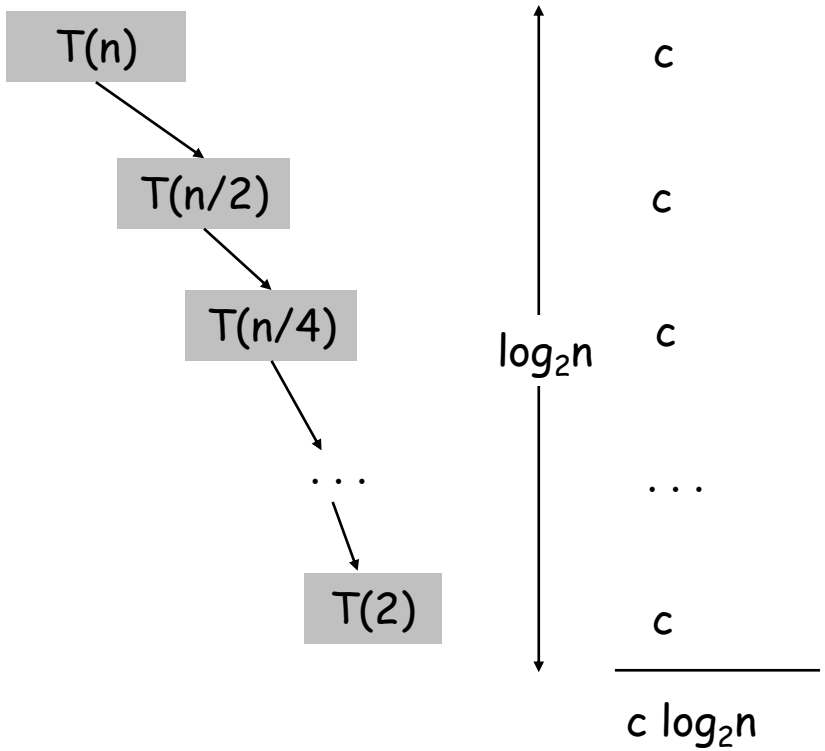
(the "sorting" slides has one slide that keeps the ceiling, so you can see that it works)

Binary Search Analysis

Binary search recurrence: $T(n) \leq c + T\left(\frac{n}{2}\right)$

Claim: The time complexity T(n) of binary search is at most c*log n

Proof 1: $T(n) \leq c + T(n/2) \leq c + c + T(n/4) \leq \ldots \leq c + c + \ldots + c$ (log n terms)



Binary Search Analysis

Binary search recurrence: $T(n) \leq c + T\left(\frac{n}{2}\right)$

Claim: The time complexity T(n) of binary search is at most c*log n

Proof 2: (induction) Base case: n=1

Now suppose that for n′ ≤ n – 1, $T(n') \leq c * \log(n')$

Then T(n) ≤ c + T(n/2) ≤ c + c*log(n/2) = c + c*(log n – 1) = c*log n

Recursive Algorithms

Exercício 2. Projete um algoritmo (recursivo) que receba como entrada um numero real x e um inteiro positivo n e devolva x^n. O algoritmo deve executar O(log n) somas e multiplicações