A First Problem: Stable Matching

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive to break current pairs

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will hook up.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

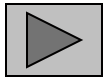
Stable roommate problem

- No gender
- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.



```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman)
    Choose such a man m
    w = favorite woman that m has not proposed to
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
```

Proof of Correctness: Termination

Observation 1. A man does not propose twice to the same woman

Claim. Algorithm terminates after at most n^2 iterations of while loop

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that not everyone is matched
  ⇒ There is a man an a woman single at termination
- Say Zeus and Amy are single at termination
- By Observation 2, Amy was never proposed to
- But, Zeus proposes to everyone, since he ends up unmatched ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. A-Z: unmatched pair in Gale-Shapley matching S*. Show it is stable

- Case 1: Z likes final partner Bertha better than A
  $\Rightarrow$ A-Z is stable.

- Case 2: Z likes A better than final partner Bertha
  $\Rightarrow$ Z proposed to A ← men propose in decreasing order of preference
  $\Rightarrow$ A rejected Z (right away or later) for better pair
  $\Rightarrow$ A prefers her final partner than Z. ← women only trade up
  $\Rightarrow$ A-Z is stable.

- In either case A-Z is stable ▪

S*

Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Natural proposal-based algorithm

Correctness guarantee. Proved that this algorithm always finds a stable matching, for any instance

Number of iterations. Proved that always takes at most n^2 iterations

Q. How long does the algorithm take? (I.e., how long does each iteration take?)

(Gale-Shapley algorithm favors men)

There may be multiple stable matchings

GS algorithm is man-optimal: Each man receives best valid partner

GS algorithm is woman-pessimal: Each woman receives worst valid partner.

(Extension: Matching Residents to Hospitals)

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

Variant 2. Unequal number of men and women.

resident A unwilling to work in Cleveland

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

(Have to update definition of stable)

Used in the US for National Resident Matching Program