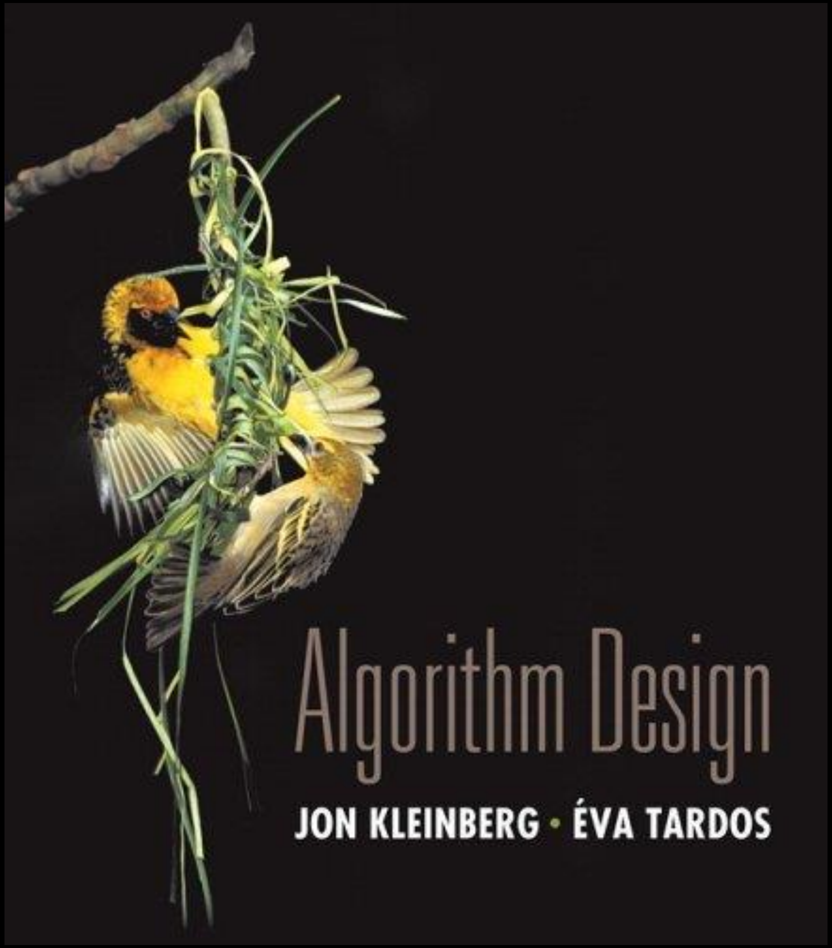


Chapter 7

Network Flow

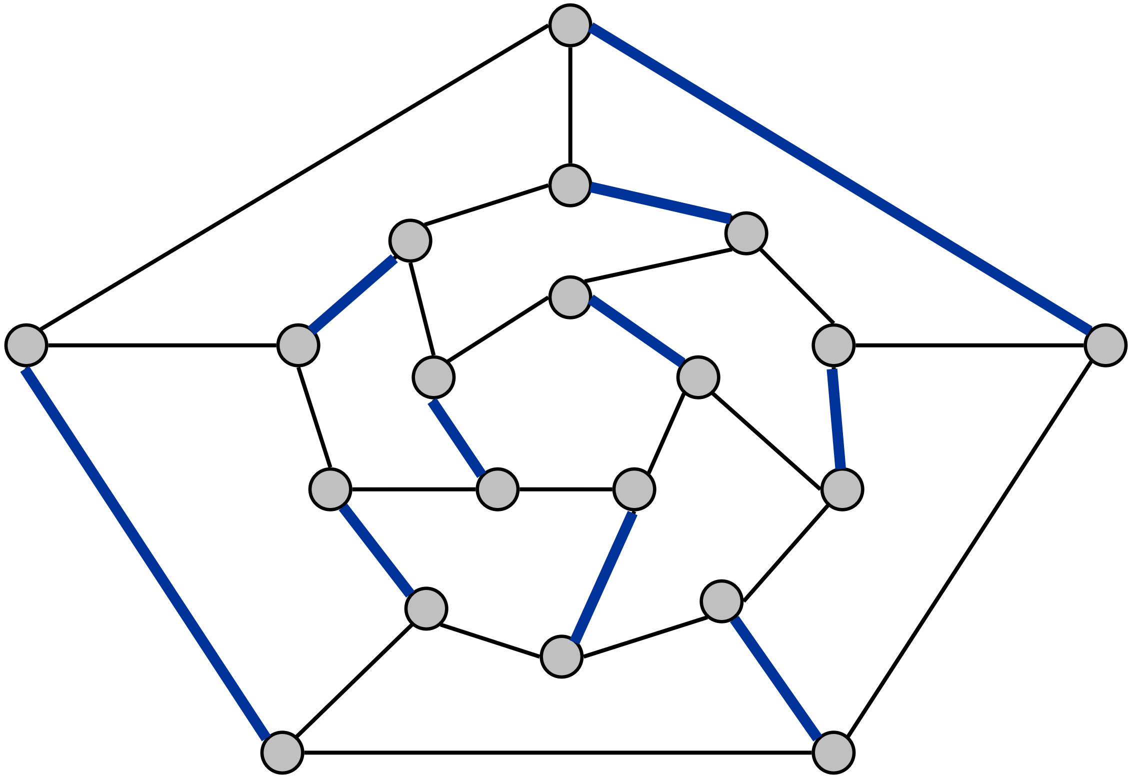
Slides by Kevin Wayne.

7.5 Bipartite Matching

Matching

Matching.

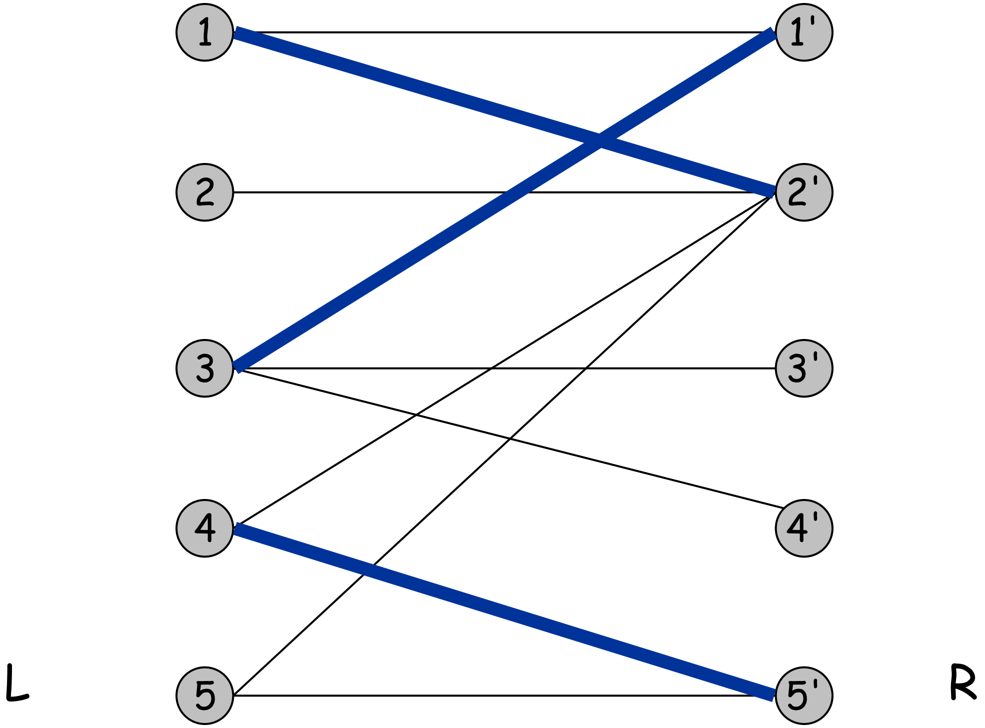
- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



matching

1-2', 3-1', 4-5'

Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
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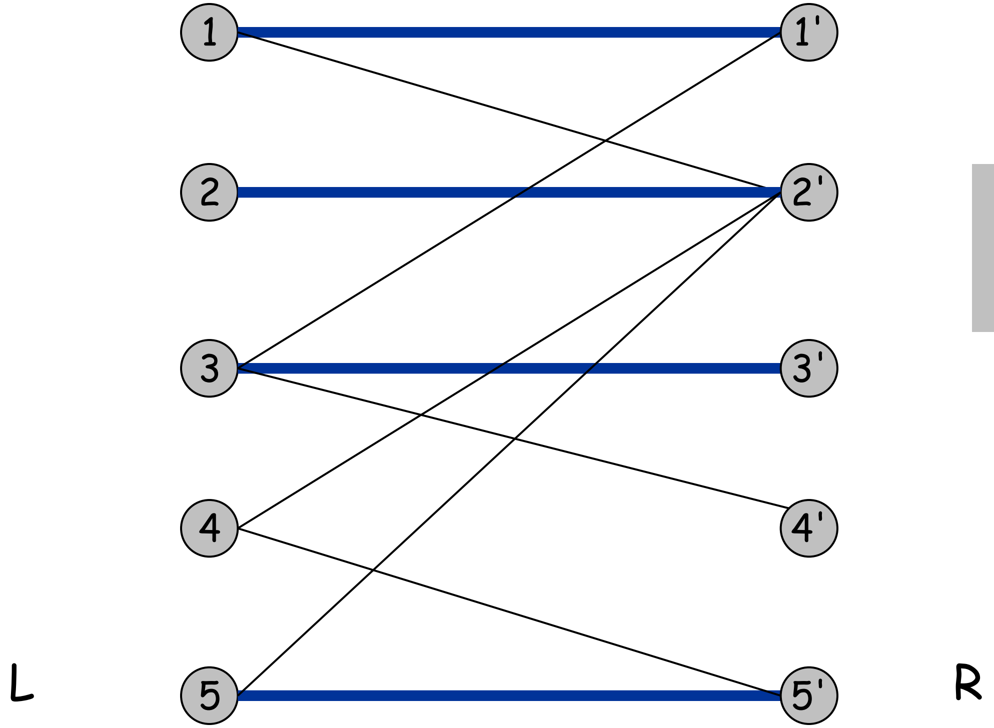


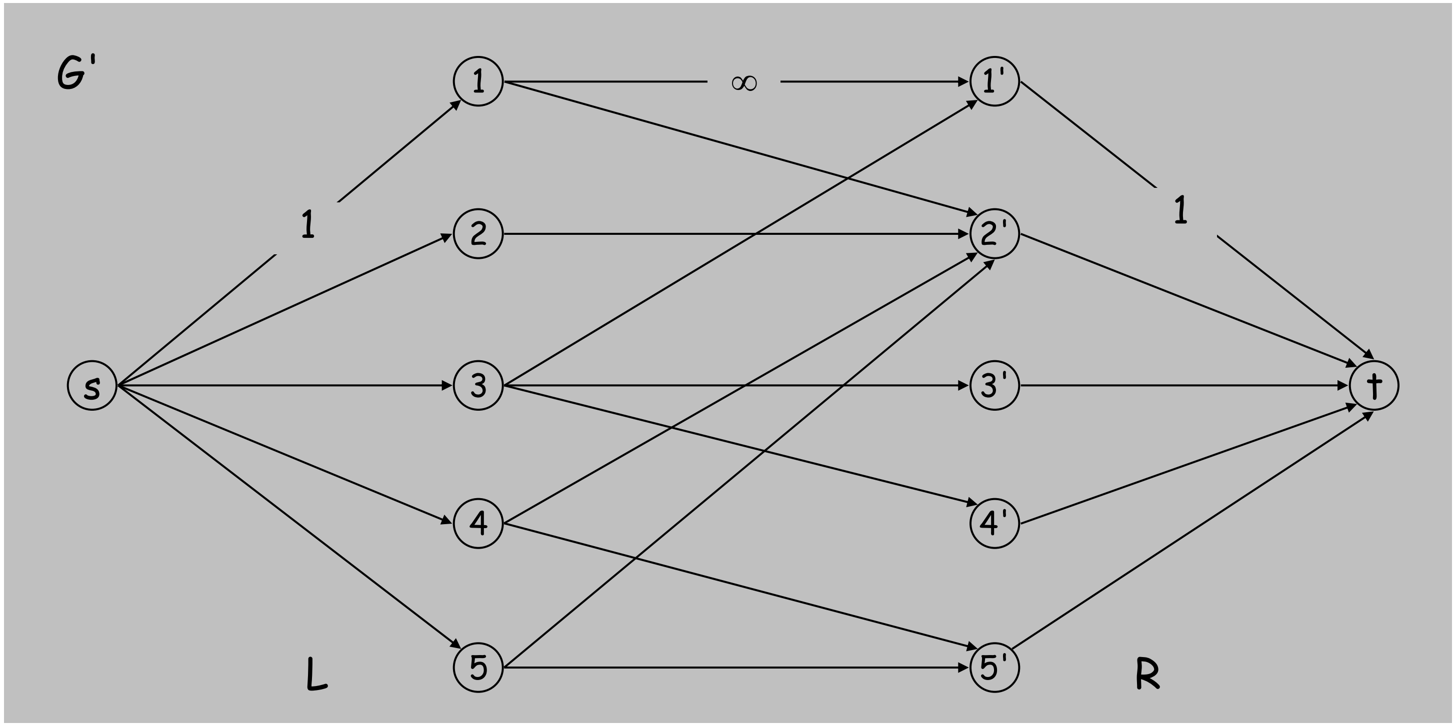
max matching

1-1', 2-2', 3-3' 4-4'

Bipartite Matching

Max flow formulation.

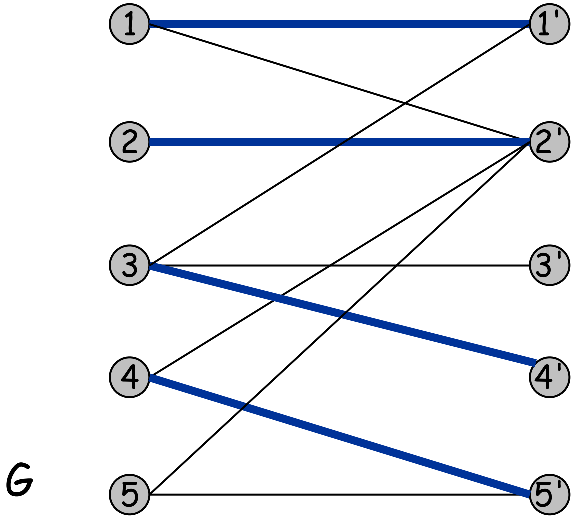
- Create digraph $G' = (L \cup R \cup \{s, t\}, \; E')$.
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.

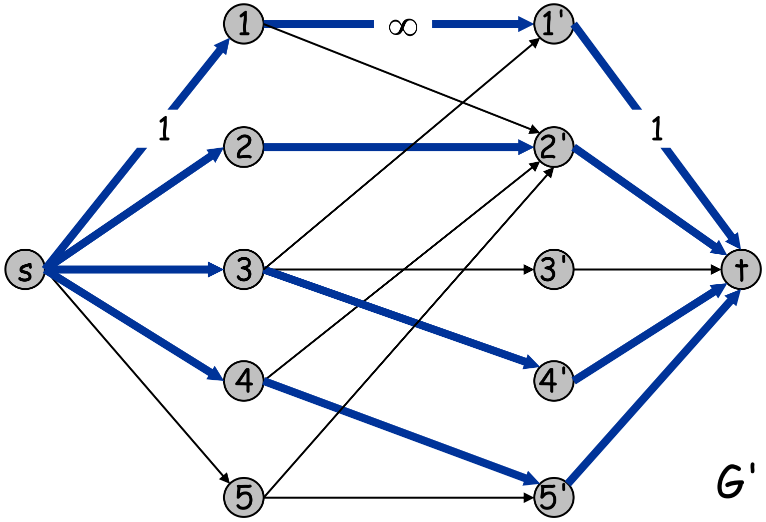


Bipartite Matching

There is a one-to-one correspondence between matchings in orig graph and integral flows in new graph

Theorem. Max cardinality matching in G = value of max flow in G'.





So we can reduce the problem of max bipartite matching to the max flow problem that we know how to solve: To solve max bipartite matching

- Create the new graph G'
- Solve the max flow problem in G'
- From this max flow extract a matching for G

Bipartite Matching: Running Time

Q: How long does it take if we use Ford-Fulkerson to solve max bipartite matching?

A: Max flow <= n, so $O(n(n+m))$

If we use other flow algorithms:

- Capacity scaling: $O(m^2 \log C) = O(m^2)$.
- Shortest augmenting path: $O(m\, n^{1/2})$.

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\, n^{1/2})$. [Micali-Vazirani 1980]

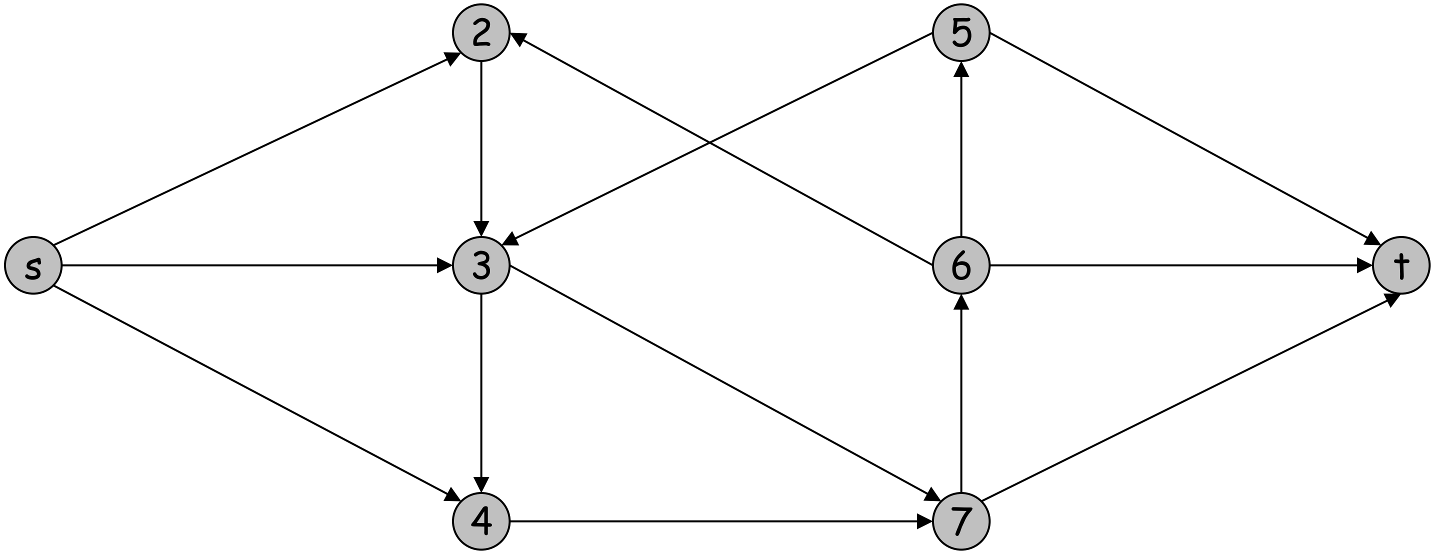
7.6 Disjoint Paths

Edge Disjoint Paths

Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s-t paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.



Edge Disjoint Paths

Disjoint path problem. Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.

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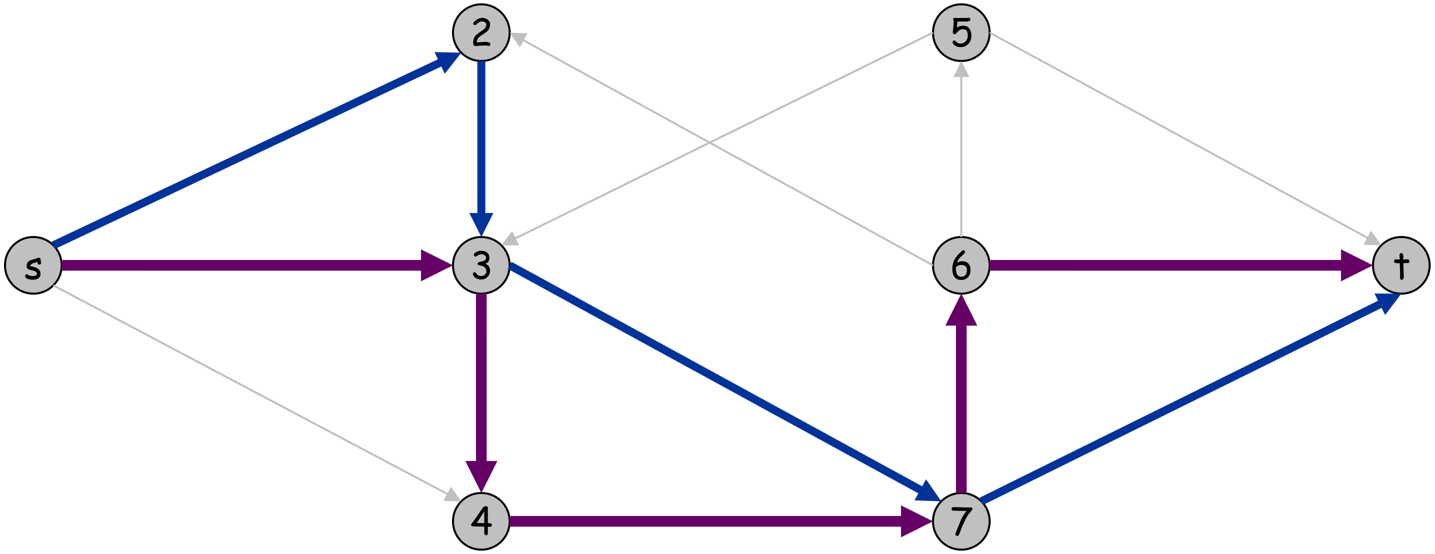
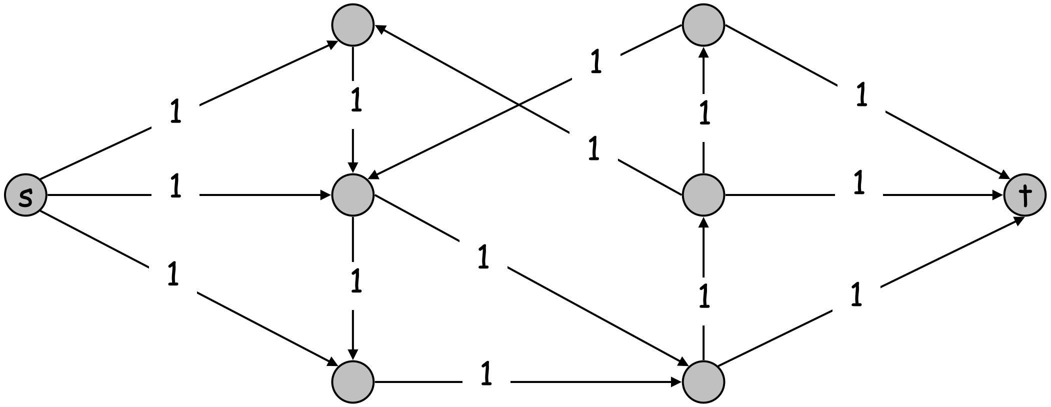
Ex: communication networks.



Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



When we decompose flow into integral flow-paths, get one-to-one correspondence with disjoint paths

- Integral flow-paths will send 1 unit of flow from s to t, and flow-paths cannot share edges because edge capacity is only 1

7.7 Extensions to Max Flow

Circulation with Demands

Circulation with demands.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

↑
demand if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$

Def. A circulation is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem: Is there a way to send supply to demand within the capacity of the network?

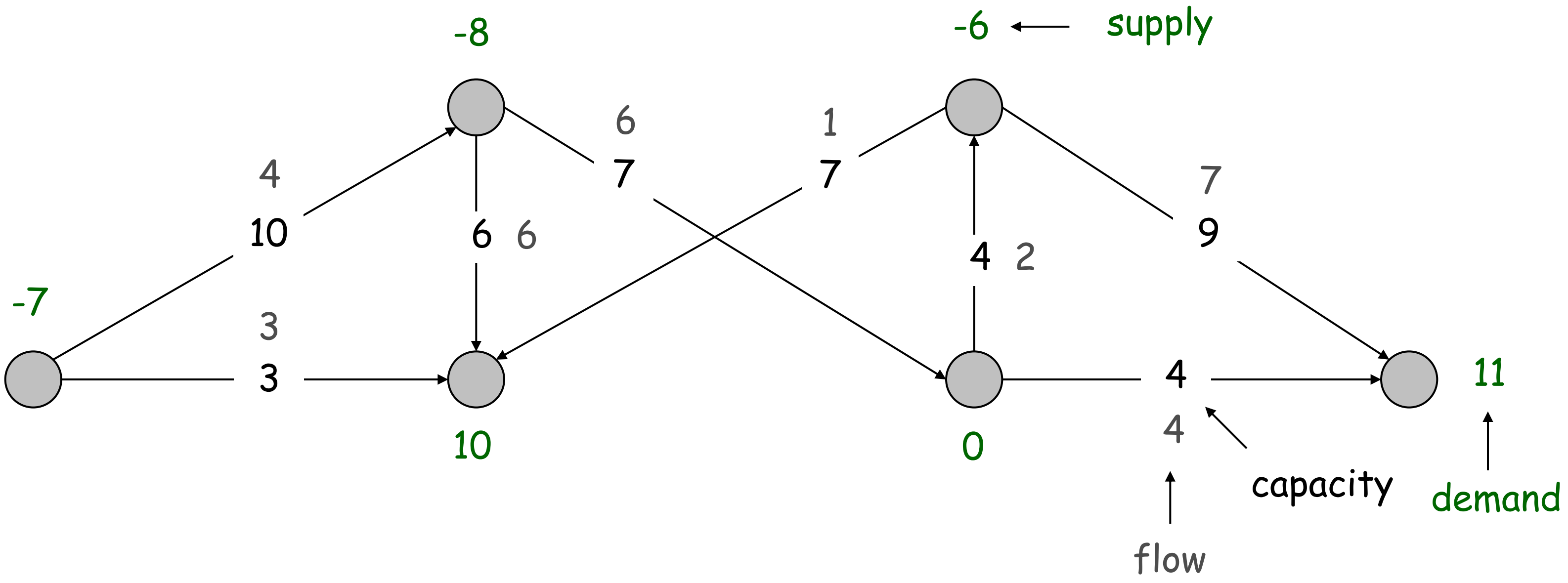
(i.e. given (V, E, c, d), does there exist a circulation?)

Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

$$\sum_{v : d(v) > 0} d(v) = \sum_{v : d(v) < 0} -d(v) =: D$$

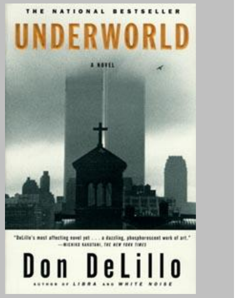
Pf. Sum conservation constraints for every demand node v.



7.12 Baseball Elimination

"See that thing in the paper last week about Einstein? . . . Some reporter asked him to figure out the mathematics of the pennant race. You know, one team wins so many of their remaining games, the other teams win this number or that number. What are the myriad possibilities? Who's got the edge?"
"The hell does he know?"
"Apparently not much. He picked the Dodgers to eliminate the Giants last Friday."

- Don DeLillo, Underworld



Baseball Elimination

Team i	Wins w_i	Losses l_i	To play r_i	Against = r_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.
- $w_i + r_i < w_j \Rightarrow$ team i eliminated.
- Only reason sports writers appear to be aware of.
- Sufficient, but not necessary!

Baseball Elimination

Team i	Wins w_i	Losses l_i	To play r_i	Against = r_{ij}			
				Atl	Phi	NY	Mon
Atlanta	83	71	8	-	1	6	1
Philly	80	79	3	1	-	0	2
New York	78	78	6	6	0	-	0
Montreal	77	82	3	1	2	0	-

Which teams have a chance of finishing the season with most wins?

- Philly can win 83, but still eliminated . . .
- If Atlanta loses a game, then some other team wins one.

Remark. Answer depends not just on how many games already won and left to play, but also on whom they're against.

Baseball Elimination

Baseball elimination problem.

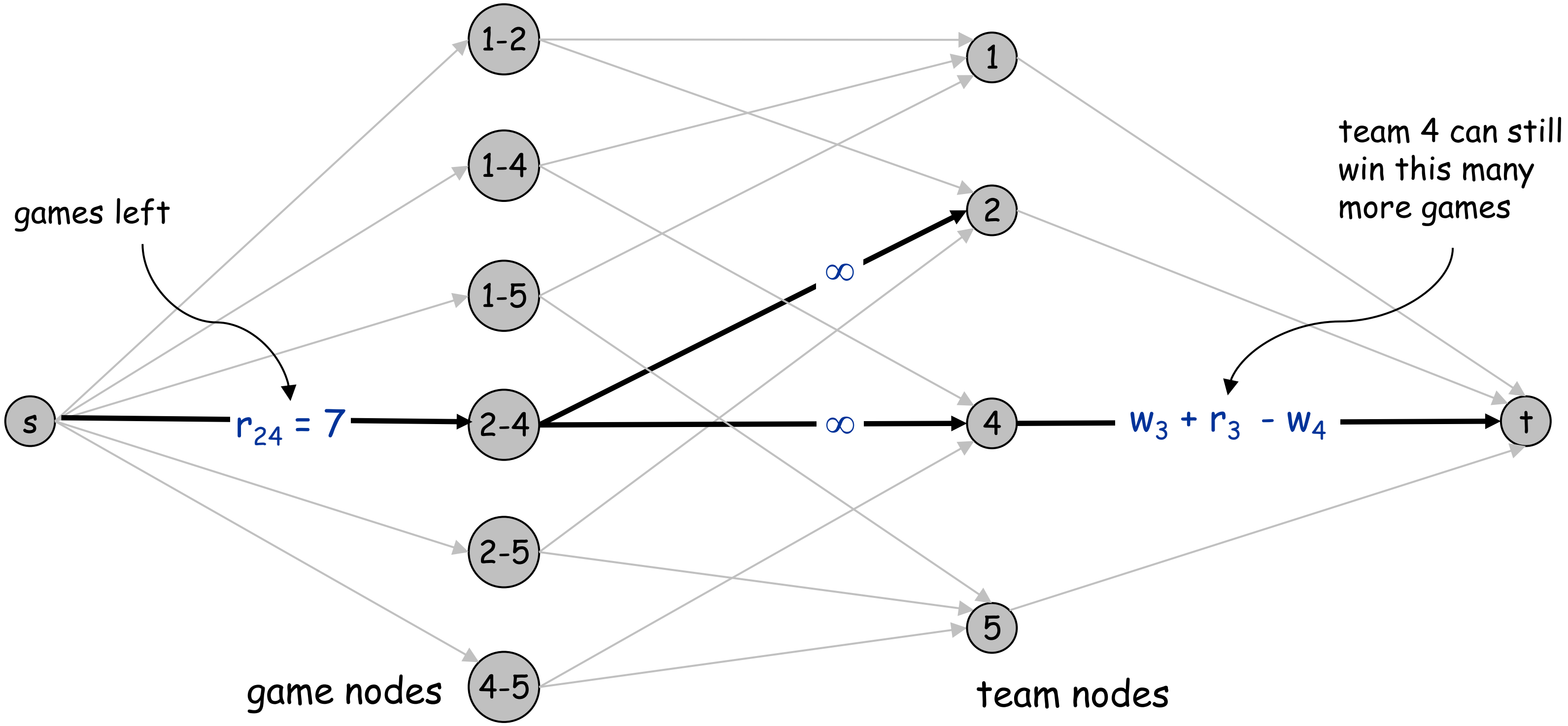
- Set of teams S.
- Distinguished team $s \in S$.
- Team x has won w_x games already.
- Teams x and y play each other r_{xy} additional times.
- Is there any outcome of the remaining games in which team s finishes with the most (or tied for the most) wins?

Let's simplify the problem: Given a fixed team, can this team finish with the most wins?

Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

- Assume team 3 wins all remaining games $\Rightarrow$ $w_3 + r_3$ wins.
- Remove all games of tem 3 from consideration....
- ... and try to divide all remaining games so that all teams have $\leq$ $w_3 + r_3$ wins.



Team 3 can finish with most wins iff max flow saturates all edges from s

Baseball Elimination: Max Flow Formulation

Theorem. Team 3 is not eliminated iff max flow saturates all edges leaving source.

- Integrality theorem ⇒ each remaining game between x and y added to number of wins for team x or team y.
- Capacity on (x, t) edges ensure no team wins too many games.