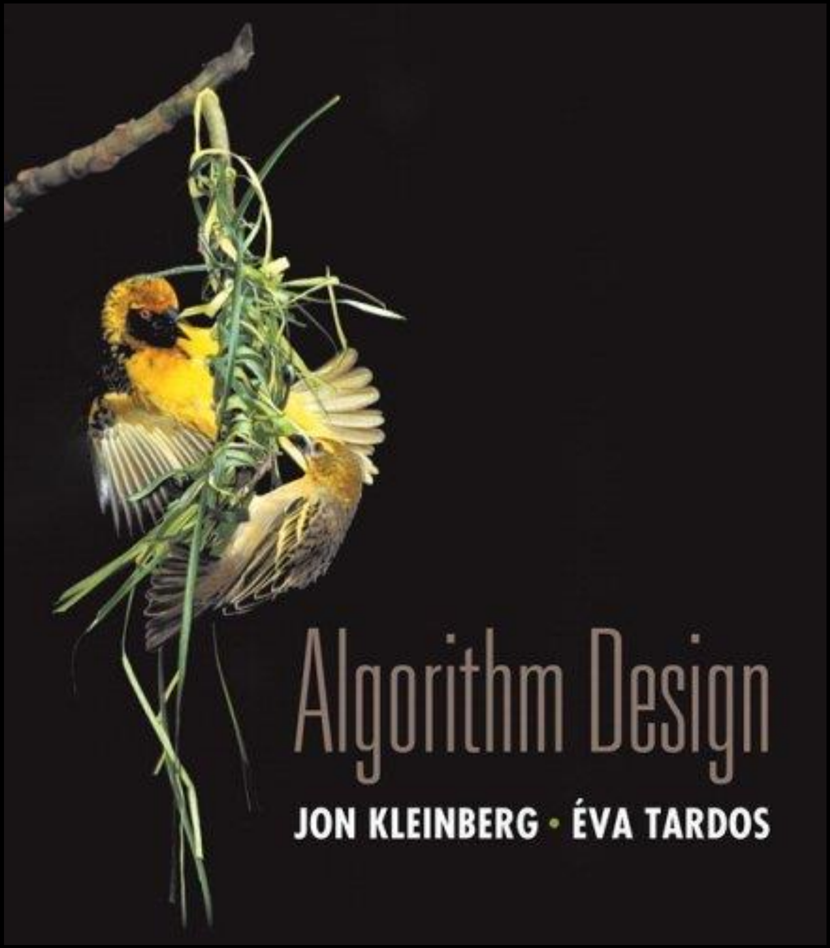


Chapter 8

NP and Computational Intractability

Slides by Kevin Wayne.

Algorithm Design Patterns and Anti-Patterns

Algorithm design patterns.	Ex.
Greed.	O(n log n) interval scheduling.
Divide-and-conquer.	O(n log n) closest points.
Dynamic programming.	$O(n^2)$ edit distance.
Linear Programming	
Reductions.	

Algorithm design anti-patterns.	
NP-completeness.	$O(n^k)$ algorithm unlikely.
Undecidability.	No algorithm possible.

8.1 Polynomial-Time Reductions

Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

A working definition. [Cobham 1964, Edmonds 1965, Rabin 1966]
Those with polynomial-time algorithms.

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover

Primality testing	Factoring

Polynomial-Time Reduction

Suppose we could solve Y in polynomial-time. What else could we solve in polynomial time?

Reduction. Problem X polynomial reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to algorithm that solves problem Y.

Notation. $X \leq_P Y$.

Remarks.

- We pay for time to write down instances sent to black box $\Rightarrow$ instances of Y must be of polynomial size.

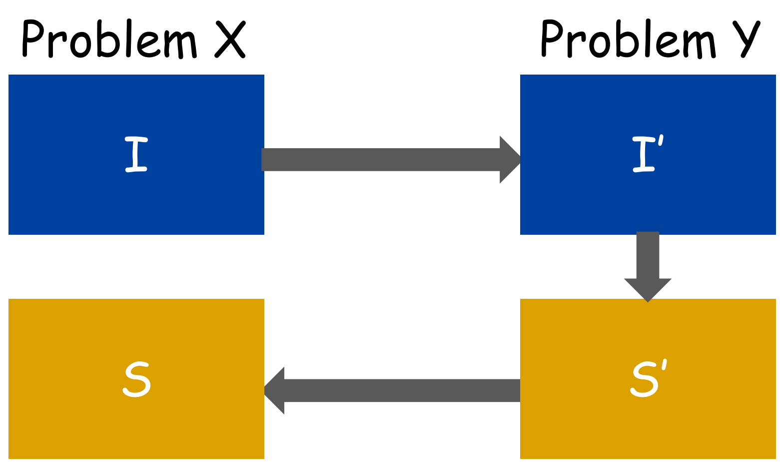
Polynomial-Time Reduction

Common way to establish polynomial time reduction from X to Y

1. Convert the instance I of problem X into an instance I' for problem Y in polynomial-time on the size of I

2. Obtain a solution S' for I' through a call to an algorithm that solves problem Y.

3. Convert the solution S' of instance I' into a solution S of instance I in polynomial-time on the size of I



Polynomial-Time Reduction

Purpose. Classify problems according to relative difficulty.

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial-time, then X can also be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Establish equivalence. If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$.

up to cost of reduction

Reduction By Simple Equivalence

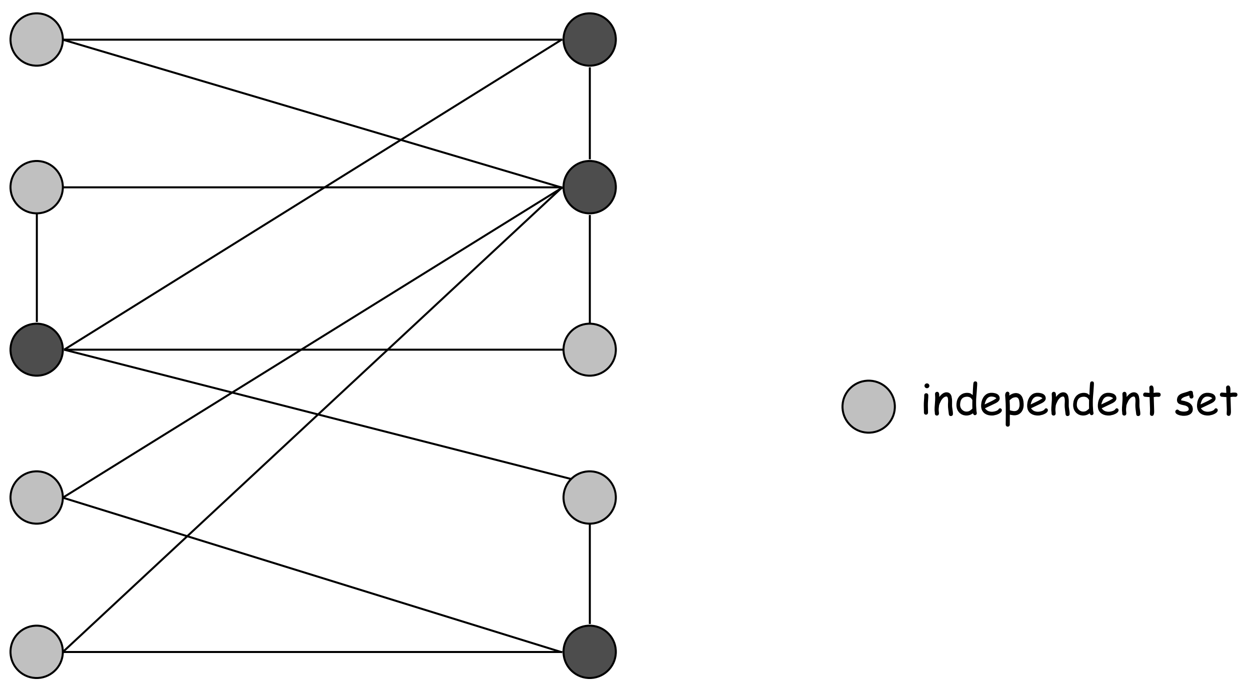
Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

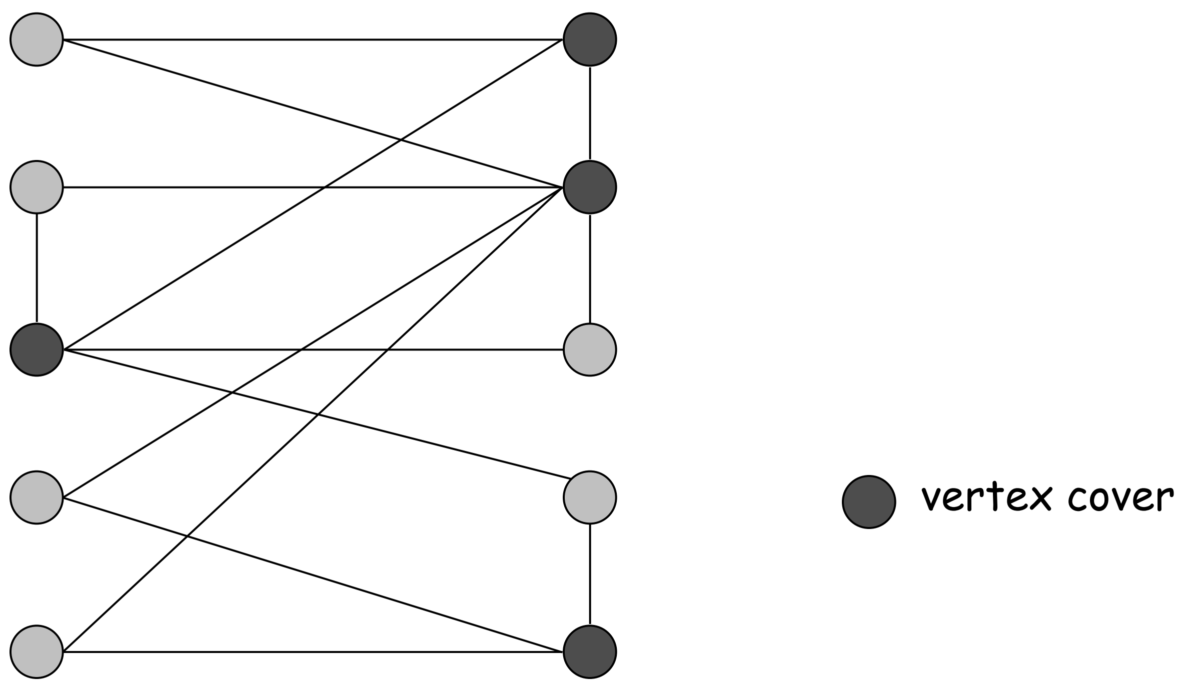
Ex. Is there an independent set of size ≥ 6? Yes.
Ex. Is there an independent set of size ≥ 7? No.



Vertex Cover

VERTEX COVER: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

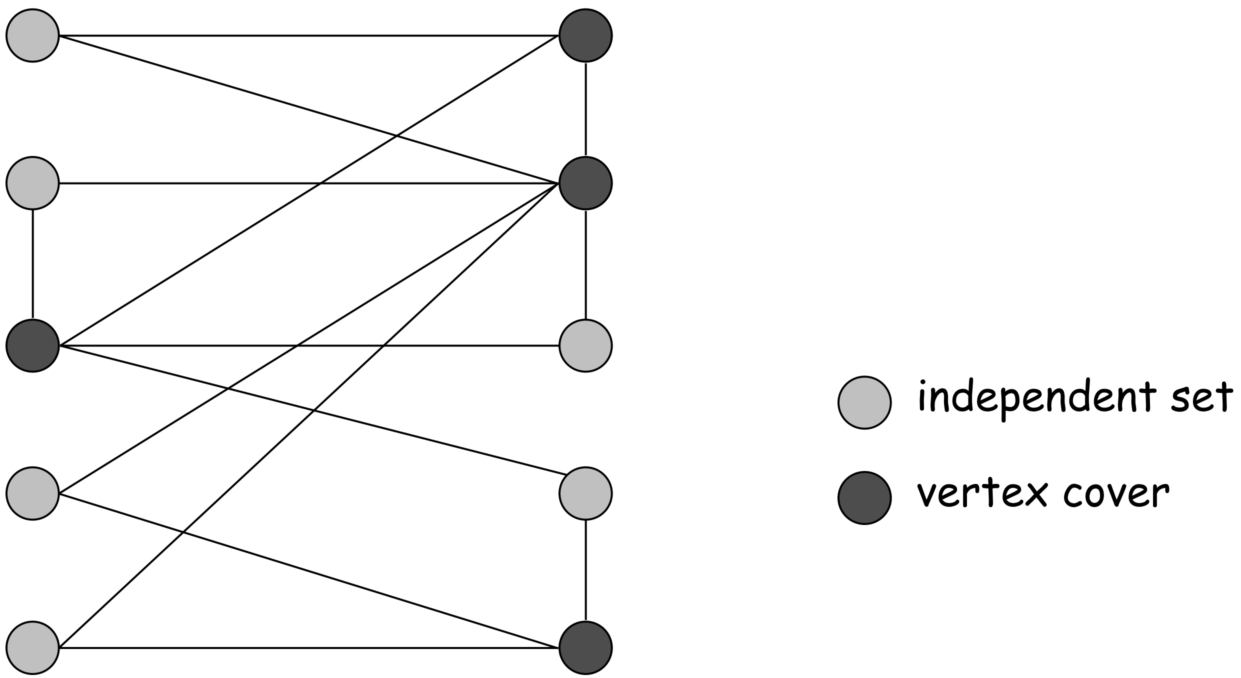
Ex. Is there a vertex cover of size ≤ 4? Yes.
Ex. Is there a vertex cover of size ≤ 3? No.



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V − S is a vertex cover.



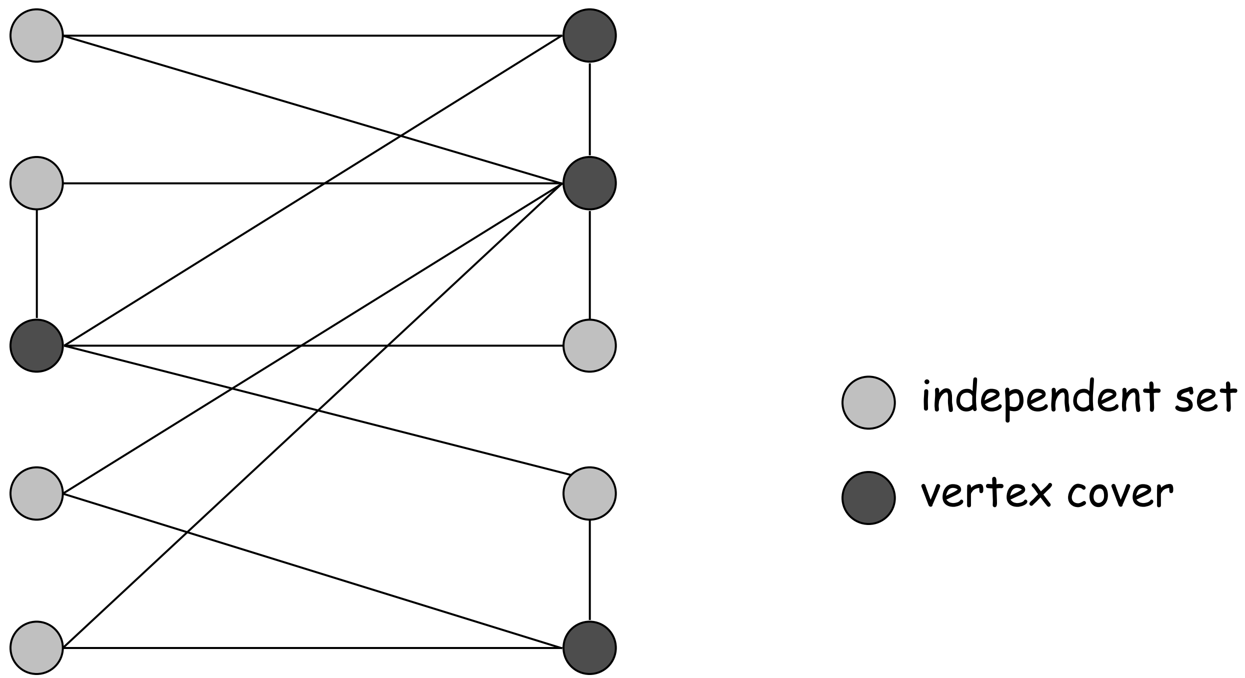
Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow$ u $\notin$ S or v $\notin$ S $\Rightarrow$ u $\in$ V – S or v $\in$ V – S.
- Thus, V – S covers (u, v).



Vertex Cover and Independent Set

Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

$\Rightarrow$

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow$ $u \notin S$ or $v \notin S$ $\Rightarrow$ $u \in V - S$ or $v \in V - S$.
- Thus, V – S covers (u, v).

$\Leftarrow$

- Let V – S be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since V – S is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

Independent Set to Vertex Cover

Solving INDEPENDENT SET using VERTEX COVER

1. Transform the instance (G,k) for Independent Set into the instance (G,n-k) for Vertex Cover

2. Call VC algorithm to solve (G,n-k)

3. If the algorithm answer YES for VC then answer YES for Independent Set. Otherwise, answer NO.

Reduction from Special Case to General Case

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq$ k of these sets whose union is equal to U?

Ex:

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$k = 2$

$S_1 = \{3, 7\}$ $\quad$ $S_4 = \{2, 4\}$

$S_2 = \{3, 4, 5, 6\}$ $\quad$ $S_5 = \{5\}$

$S_3 = \{1\}$ $\quad$ $S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

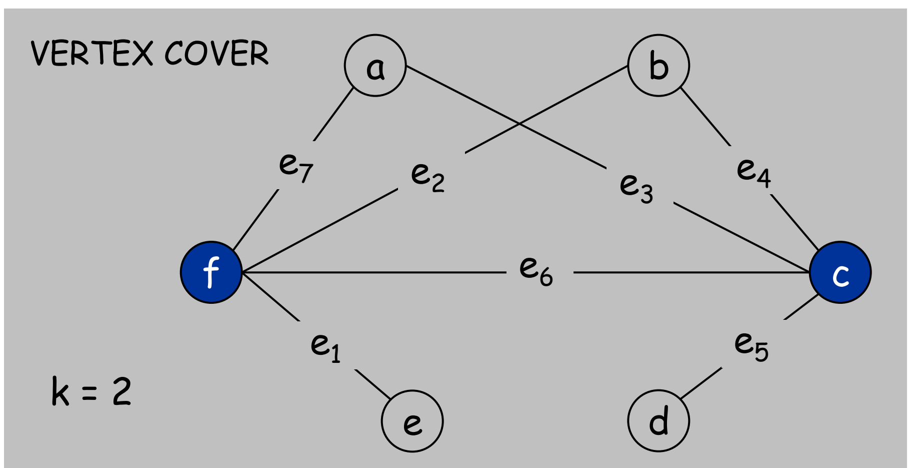
Claim. VERTEX-COVER $\le_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance (G, k), we construct a SET-COVER instance with one-to-one correspondence between solutions

Construction.

- Create SET-COVER instance:

  $k = k, \;\; U = E, \;\; S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size $\le k$ iff vertex cover of size $\le k$. ▪



SET COVER

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$k = 2$
$S_a = \{3, 7\}$ $\quad S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $\quad S_d = \{5\}$
$S_e = \{1\}$ $\quad S_f = \{1, 2, 6, 7\}$

8.2 Reductions via "Gadgets"

Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

Satisfiability

3-CNF: A formula on boolean variables of the form "AND's of [OR's of three variables or their complements]"

3-SAT Problem: Given a 3-CNF, is there a setting of the variables that makes the formula TRUE? (i.e. is the formula *satisfiable*?)

Ex: $$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3 \vee x_4\right) \wedge \underbrace{\left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)}_{\text{clause}}$$

Yes: x_1 = true, x_2 = true x_3 = false.

3 Satisfiability Reduces to Independent Set

Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT with k clauses, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3 Satisfiability Reduces to Independent Set

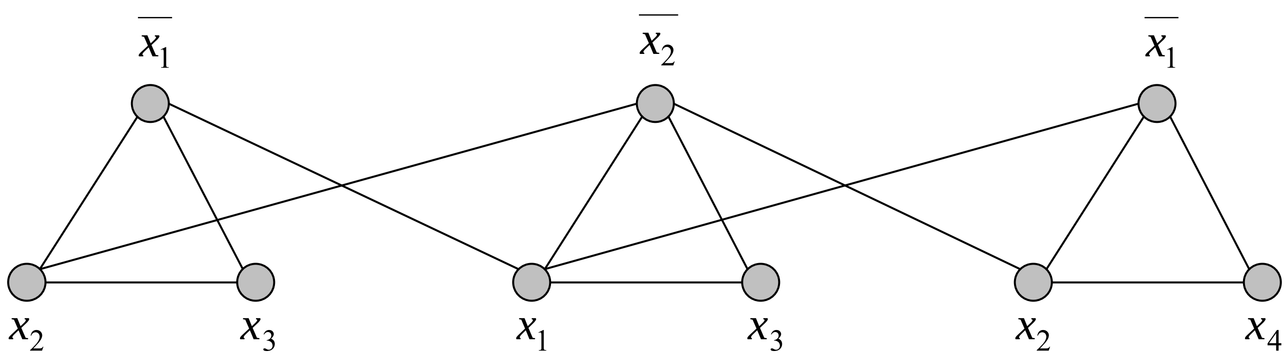
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT with k clauses, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

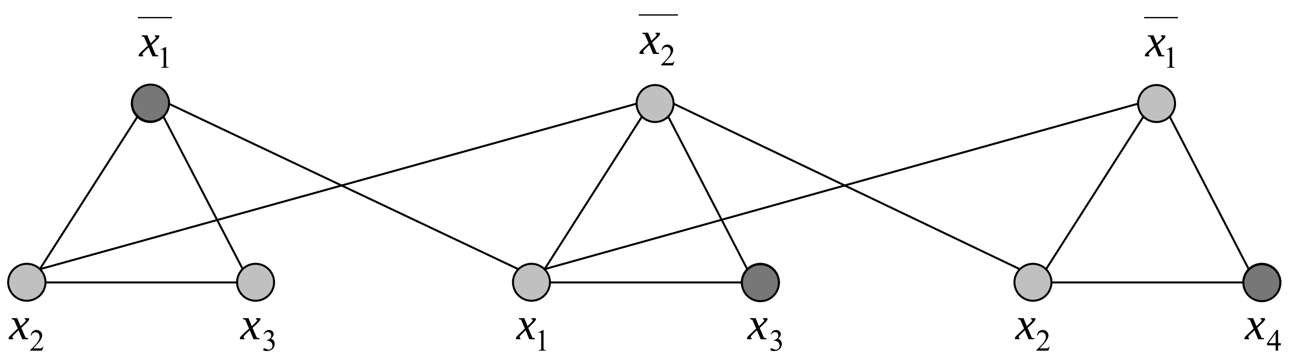
3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. $\leftarrow$ and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3 Satisfiability Reduces to Independent Set

Claim. G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

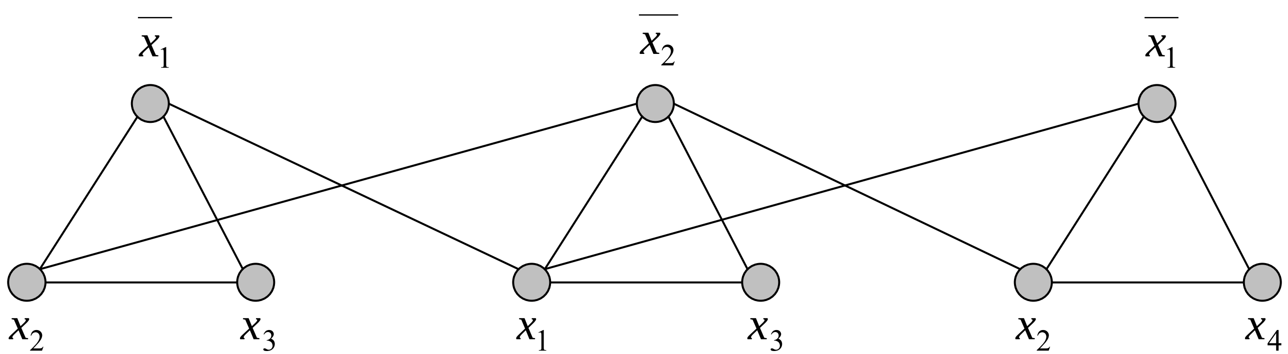
Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. ← and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Suppose the formula has a satisfying assignment

- Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.

Pf idea. Compose the two algorithms.

Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Self-Reducibility

Decision problem. Does there exist a vertex cover of size $\leq$ k?
Search problem. Find vertex cover of minimum cardinality.

Clear: decision version $\leq_P$ search version

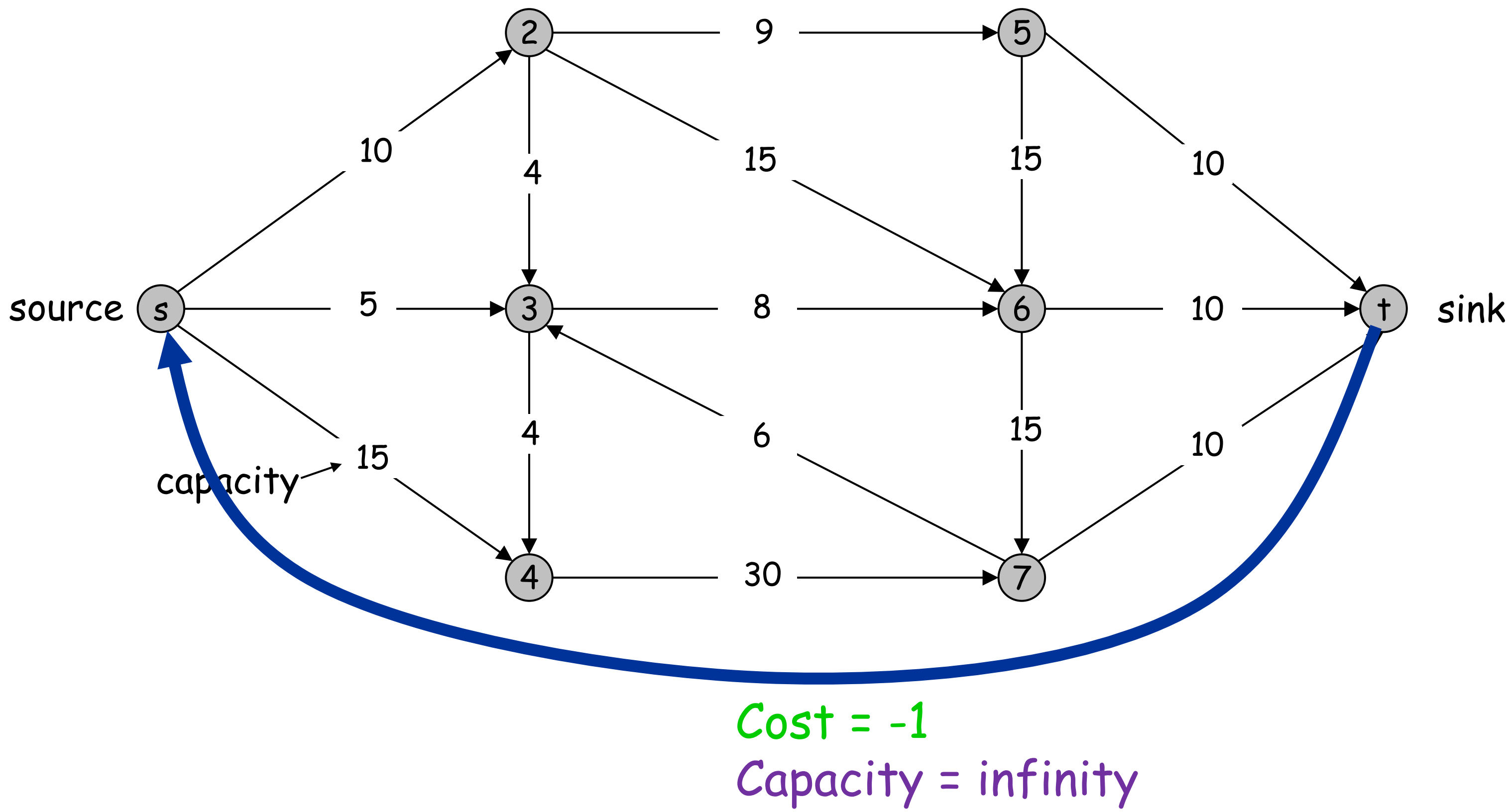
Self-reducibility. Search problem $\leq_P$ decision version.
- Applies to all (NP-complete) problems in this chapter.
- Justifies our focus on decision problems.

Ex: to find min cardinality vertex cover.
- (Binary) search for cardinality k* of min vertex cover.
- Find a vertex v such that $G - \{ v \}$ has a vertex cover of size $\leq$ k* - 1.
  - any vertex in any min vertex cover will have this property
- Include v in the vertex cover.
- Recursively find a min vertex cover in $G - \{ v \}$.

delete v and all incident edges

Exercise

Exercise: Reduce the Max-flow problem to the Min-cost flow problem



Add edge from sink to source with negative cost and infinite capacity (put cost 0 on all other edges), all demands/supply equal to 0

8.3 Definition of NP

Decision Problems

Decision problem.

- X is a set of strings.
- Instance: string s.
- Algorithm A solves problem X: A(s) = `yes` iff $s \in X$.

Polynomial time. Algorithm A runs in poly-time if for every string s, A(s) terminates in at most p(|s|) "steps", where p(·) is some polynomial.

length of s

PRIMES: X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, …. }
Algorithm. [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

INDEPENDENT SET :
X = { family of graph with an independent set of size at least k}
Algorithm. A polynomial algorithm is not known

Definition of P

P. Decision problems for which there is a poly-time algorithm.

The decision version of most problems we saw on the course are in P

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

NP

NP (informal). Problems that can be certified in poly-time

Certification algorithm intuition.

- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s, t) =$ `yes`.

"certificate" or "witness"

NP. Decision problems for which there exists a poly-time certifier.

$C(s, t)$ is a poly-time algorithm and $|t| \le p(|s|)$ for some polynomial $p(\cdot)$.

Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate.

Certifier.

Ex.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

Ex.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

$$x_1 = 1, \; x_2 = 1, \; x_3 = 0, \; x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

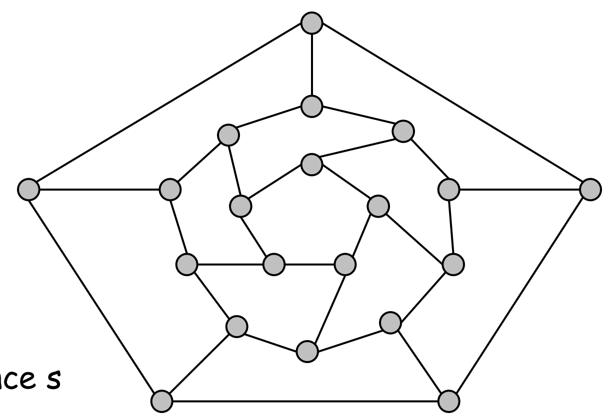
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?

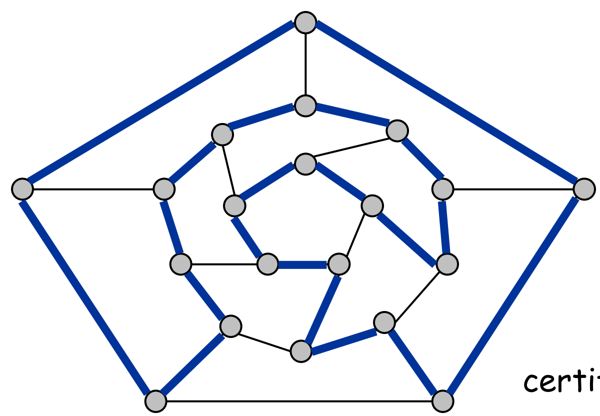
Certificate.

Certifier.

Conclusion. HAM-CYCLE is in NP.



instance s



certificate t

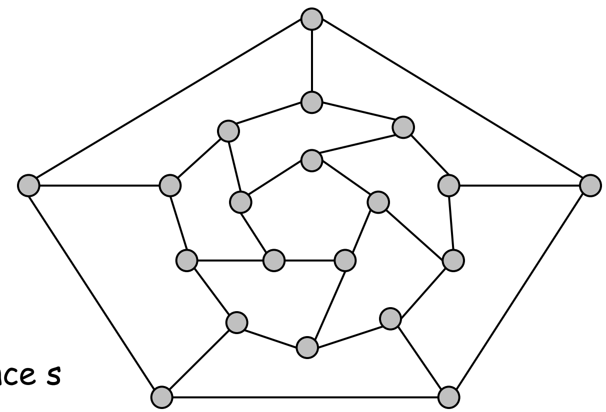
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

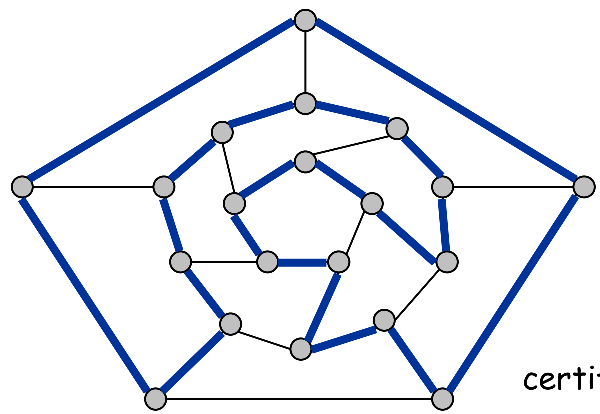
Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



instance s



certificate t

Certifiers and Certificates: COMPOSITE

COMPOSITE. Is x a composite number ?

Certificate. Two factors of x larger than 1.

Certifier. Verify if the two number are larger than 1 and if their product is x

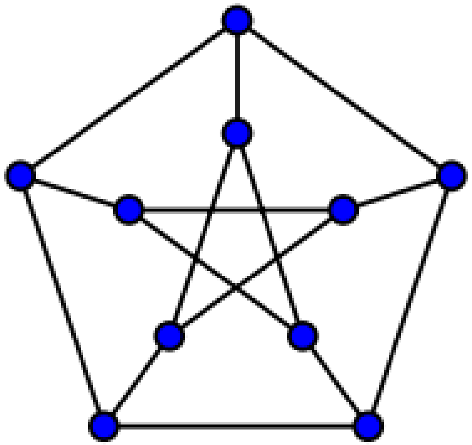
Conclusion. COMPOSITE is in NP.

Certifiers and Certificates: Non-Hamiltonian

Non-Hamiltonian. Is the graph G non-Hamiltonian?

Certificate.

It is believed that this problem is not in NP

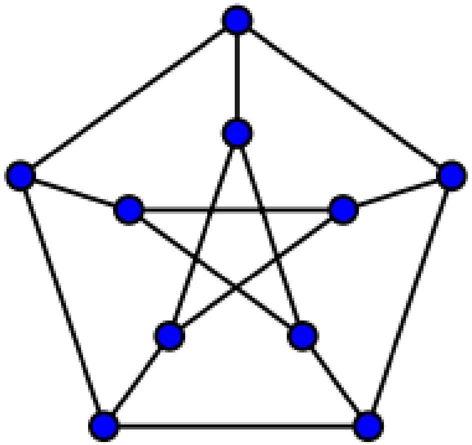


Certifiers and Certificates: Non-Hamiltonian

Non-Hamiltonian. Is the graph G non-Hamiltonian?

Certificate. ???

It is believed that this problem is not in NP



Certifiers and Certificates: PRIMES

PRIME. Is x a prime number ?

Certificate. empty

Certifier. AKS primality algorithm that runs in polynomial time

Conclusion. PRIMES is in NP.

The same reasoning allows us to conclude that P $\subseteq$ NP
(that is, every problem with a polynomial-time algorithm
can be certified in polynomial time)

P, NP, EXP

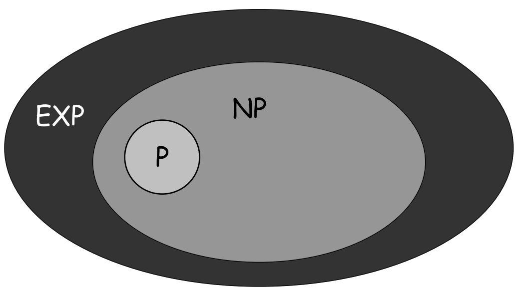
P. Decision problems for which there is a poly-time algorithm.
EXP. Decision problems for which there is an exponential-time algorithm.
NP. Decision problems for which there is a poly-time certifier.

Claim. P $\subseteq$ NP $\subseteq$ EXP.

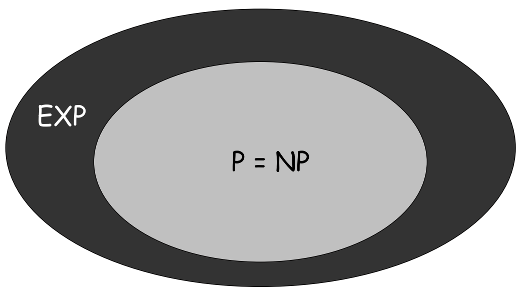
The Main Question: P Versus NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
(is finding the solution as easy as checking the solution?)

Clay $1 million prize.



If $P \neq NP$



If $P = NP$

would break RSA cryptography
(and potentially collapse economy)

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

Consensus opinion on P = NP? Probably no.

Progress on P vs NP?

Some ways of proving P!=NP do not work:

- Relativization [Baker, Gill, Solovay '75]
- Natural Proofs [Rudich, Razborov '97]
- Algebrization [Aaronson, Wigderson '08]

Some ways of proving P=NP do not work:

- Extended formulations [Yannakakis '91, Fiorini et al. '12, Rothvoss '14]

Maybe cannot be proved using standard math (i.e. independent of ZFC)

- If P!=NP, some independence proofs would not work [Ben-David, Halevi '92]

8.4 NP-Completeness

NP-Complete

"Hardest" NP problems

NP-complete. Problem Y is *NP-complete* if is it in NP and for every problem X in NP, $X \leq_p Y$.

Not clear they exist, could have harder and harder problems in NP

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.

Pf. $\Leftarrow$ If P = NP then Y can be solved in poly-time since Y is in NP.

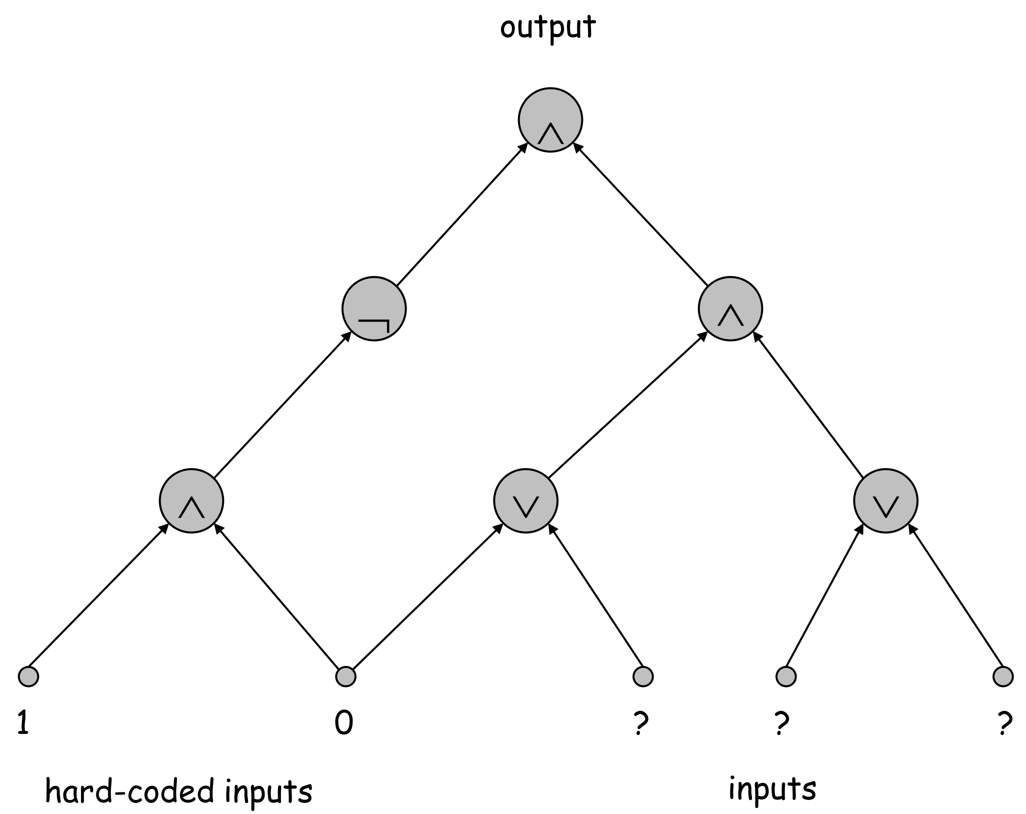
Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.

- Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies NP $\subseteq$ P.
- We already know P $\subseteq$ NP. Thus P = NP. ▪

Circuit Satisfiability

CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1



Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

The "First" NP-Complete Problem

Theorem. CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

Pf. (sketch)

- Any algorithm that takes a fixed number n of bits as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

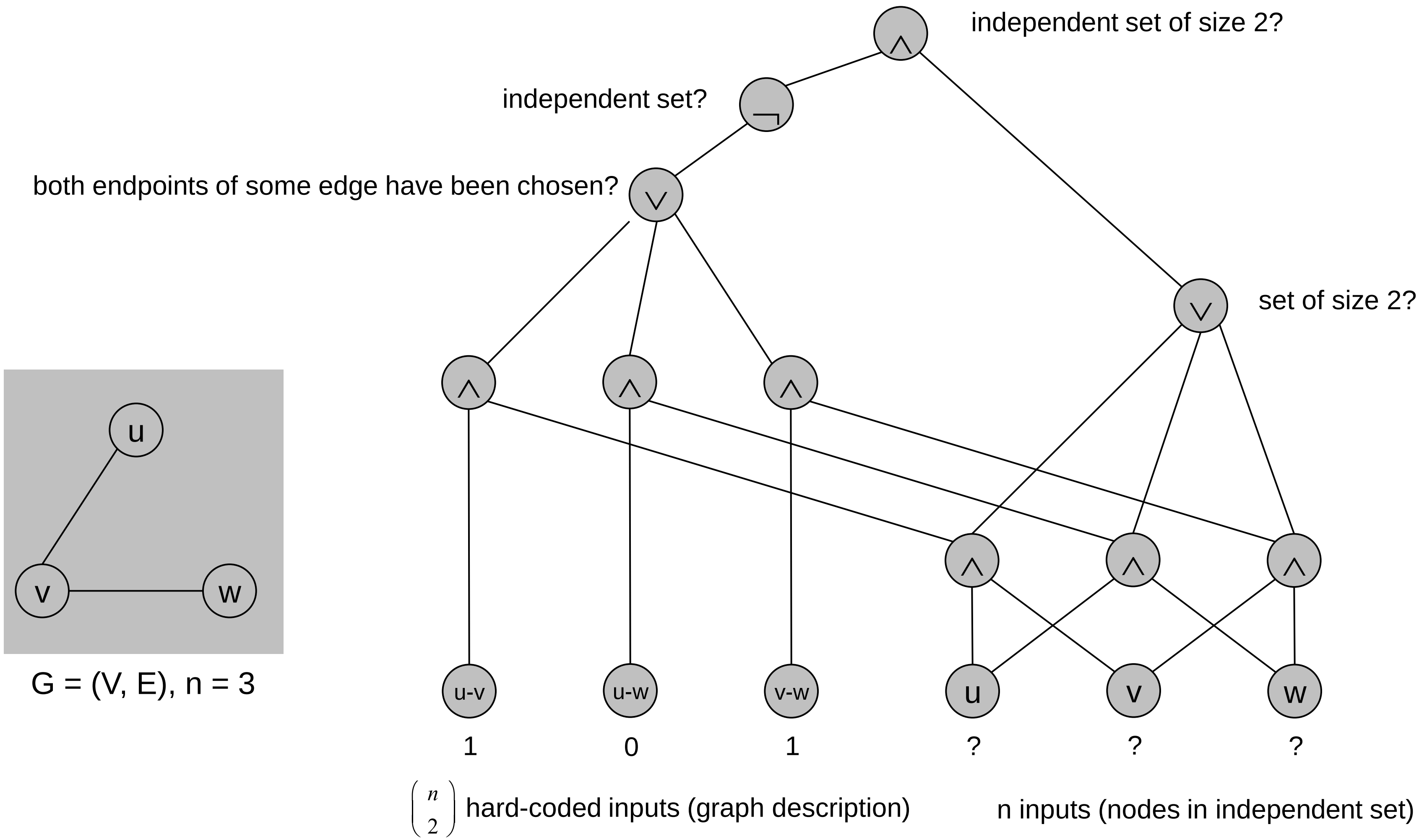
- Consider some problem X in NP. It has a poly-time certifier C(s, t). To determine whether s is in X, need to know if there exists a certificate t of length p(|s|) such that C(s, t) = `yes`

- View C(s, t) as an algorithm on |s| + p(|s|) bits (input s, certificate t) and convert it into a poly-size circuit K.
  - first |s| bits are hard-coded with s
  - remaining p(|s|) bits represent bits of t

- Circuit K is satisfiable iff there is certificate t for which C(s, t) = `yes`.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that K outputs true iff graph G has an independent set of size 2.



Establishing NP-Completeness

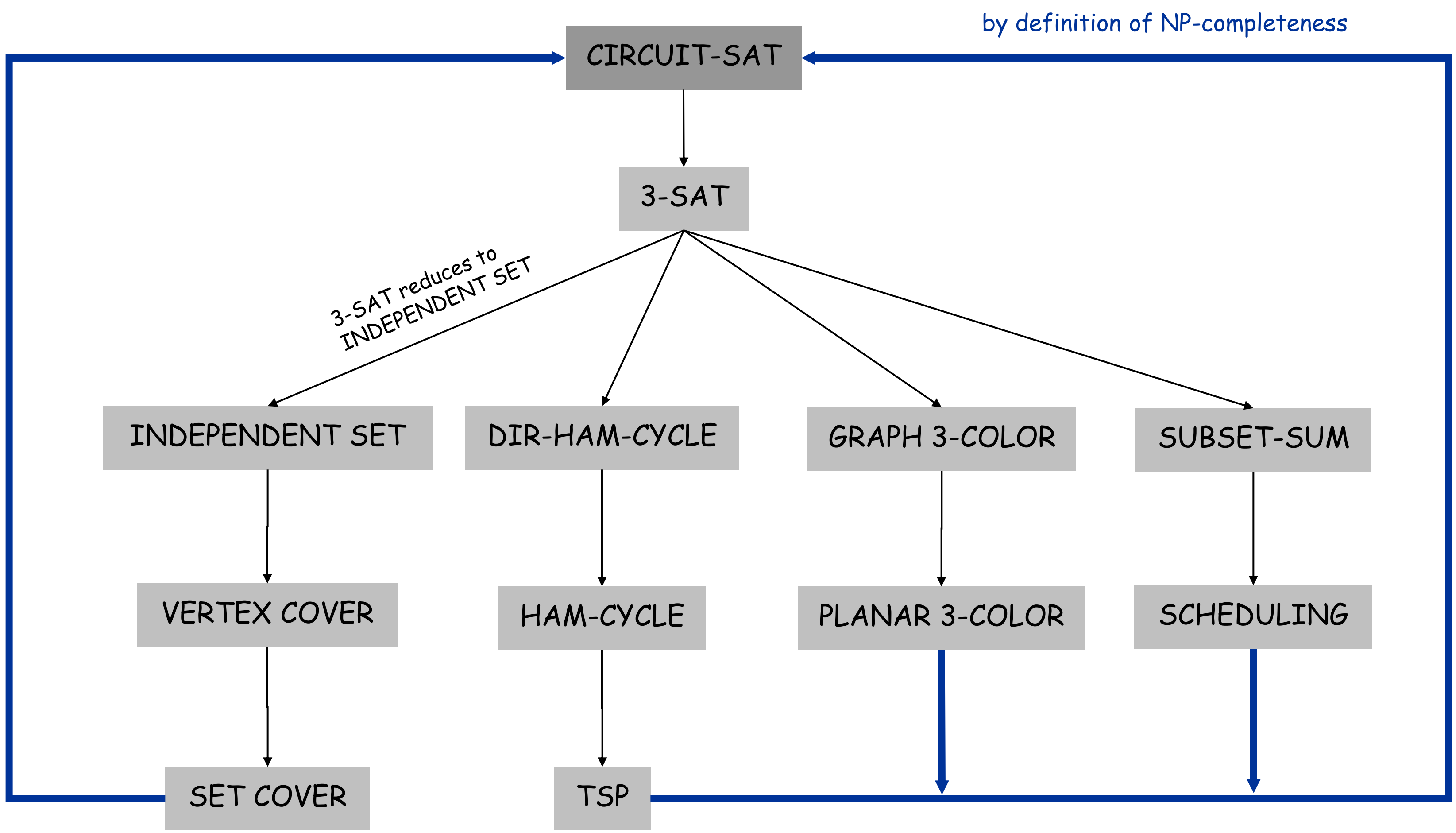
Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe to establish NP-completeness of problem Y.

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X.
- Step 3. Prove that $X \leq_p Y$. (so Y is as hard as the hardest NP problems)

NP-Completeness

Observation. All problems below are NP-complete and polynomially reduce to one another!



Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (title, abstract, keywords).
  - more than "compiler", "operating system", "database"
- Broad applicability and classification power.
- "Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

- If a problem is NP-complete, do not try to find a polynomial time algorithm
  (would solve all NP problems, many of which very smart people worked on for a long time)

Coping With NP-Completeness

Q. Suppose I need to solve an NP-complete problem. What should I do?

Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

Approaches to cope with NP-Completeness

- Approximation Algorithms:
- Exact Methods:
  - Branch and Bound, Integer Programming
- Heuristics:
  - Local Search, Simulated annealing, Genetic Algorithms
- Beyond worst-case analysis

.

Cursos Futuros

- Otimização Combinatória
  - Programação Linear
  - Algoritmos Aleatorizados
  - Algoritmos de Aproximação

- Programação Inteira

- Heurísticas/ Meta-Heurísticas