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A note on the size of minimal covers

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Abstract

For the class of monotone boolean functions $f:\{0,1\}^n \to \{0,1\}$ where all 1-certificates have size 2, we prove the tight bound $n \leqslant (\lambda+2)^2/4$, where λ is the size of the largest 0-certificate of f.

This result can be translated to graph language as follows: for every graph $G=(V,E)$ the inequality $|V| \leqslant (\lambda+2)^2/4$ holds, where λ is the size of the largest minimal vertex cover of G. In addition, there are infinitely many graphs for which this inequality is tight.
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1. Introduction

Let $f:\{0,1\}^n \to \{0,1\}$ be a monotone boolean function. A 1-certificate (0-certificate) for f is a minimal set of variables such that if all of them are set to 1 (0) then f evaluates to 1 (0) regardless of the values assigned to the other variables. Measures related to certificates play an important role in the study of monotone boolean functions [1].[1]

In [2], Simon shows that every monotone boolean function over a set of n variables has a certificate of size at least $(1/2)\log n - (1/2)\log\log n + 1/2$. In [3], Wegener shows that this bound is tight up to $O(\log\log n)$ terms for the monotone address function.

The size $\mu(f)$ of the largest certificate (the largest among the 0-certificates and 1-certificates) for f plays a role in decision tree complexity. It is known that the minimum depth of a decision tree which evaluates f is at most $\mu(f)^2$ and that this bound is tight [1]. This parameter is also important in the study of the extremal competitive ratio of monotone boolean functions, a topic that has received some attention by the theory community recently [7]. It is known that the extremal competitive ratio of a monotone boolean function f is between $\mu(f)$ and $2\mu(f)$ and it is conjectured that it is exactly $\mu(f)$ [7,5,6].

Here, we study an issue related to the size of the largest certificate of f. For the class of monotone

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[1] In fact, the concept of certificate can be extended to every boolean function.

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boolean functions where all 1-certificates have size k, the inequality $n \leqslant k\lambda^k$ holds, where λ is the size of a largest 0-certificate of f. This can be proved almost directly by using Lemma 7.3 presented on Jukna's book [4]. However, this bound is not tight.

The main result of our paper is the tight bound $n \leqslant (\lambda+2)^2/4$ for the case $k=2$. The restriction $k=2$ allows us to work with graphs, where 1-certificates can be seen as edges and 0-certificates as minimal covers. Thus, our main result can be translated to graph language as follows: for every graph $G=(V,E)$, $|V| \leqslant (\lambda+2)^2/4$, where λ is the size of a largest minimal vertex cover of G.

We note that the quadratic upper bound $|V| \leqslant \lambda+\lambda^2$ can be proved in four lines as follows. Let C be the largest minimal cover of G and let v be the vertex of C with maximum degree. Then, observe that the degree $d(v)$ of v is at most $|C|$, for otherwise any minimal cover that does not contain v would be larger than C. Thus, we have that $|V| \leqslant |C| + \sum_{x\in C} d(x) \leqslant \lambda + \lambda_2$. As a matter of fact, to obtain our main result, we need to work harder.

2. The size of minimal covers

In order to fix the notation, we present some basic definitions that are going to be used throughout this paper.

2.1. Definitions and notations

Given an undirected graph $G=(V,E)$, a cover of G is a subset V' of V such that for all edge $(u,v)\in E$ either $u\in V'$ or $v \in V'$ (or both). A *cover C* is said to be *minimal* when $\forall x\in C$, $C-x$ is not a cover. We say that a cover C of G is *largest minimal* if and only if $|C| \geqslant |S|$ for all minimal cover S of G. Observe that a largest minimal cover is not necessarily unique.

Let $X \subseteq V$ and let $x\in X$. The *external neighborhood* of x with respect to (w.r.t.) X is the set denoted by $N_X(x)$ and defined by

$$N_X(x) = \{ y \mid (x,y)\in E \text{ and } y \notin X \}.$$

The external neighborhood of a set $U \subseteq X$ w.r.t. X is denoted by $N_X(U)$ and defined by

$$N_X(U) = \bigcup_{x\in U} N_X(x).$$

Let $X\subseteq V$. The *external degree* of $x\in X$, denoted by $|N_X(x)|$, is the cardinality of $N_X(x)$, and the external degree of $U \subseteq X$ is $|N_X(U)|$.

2.2. Results

In what follows we present a lemma that is quite useful to establish our main result.

Lemma 2.1. *Let $G(V,E)$ be a simple graph and C a largest minimal cover of G. For all $x\in C$ there is a set S_x (possibly empty) that simultaneously satisfies the three conditions below*:

(1) $S_x \subseteq C - \{x\}$.
(2) $|S_x| \geqslant |N_C(x)| - 1$.
(3) $N_C(S_x) \subseteq N_C(x)$.

Proof. Let $S_1 = (C-x)\cup N_C(x)$. We have that S_1 is a cover of G (not necessarily minimal). Let $T\subseteq S_1$ be a minimal cover for G. We have $N_C(x)\subseteq T$, for otherwise one of the edges that touches x would not be covered by T. Let us define $S_x = S_1 - T$. Note that $S_x \subseteq C - x$, which satisfies condition (1).

Since C is a largest minimal cover of G, we also have that $|T| \leqslant |C|$. Thus,

$$\begin{aligned}|S_x| &= |S_1 - T| = |S_1| - |T| \\ &\geqslant |S_1| - |C| = \left|C - \{x\} \cup N_C(x)\right| - |C| \\ &= \left|N_C(x)\right| - 1.\end{aligned}$$

Hence, we can conclude that S_x also satisfies condition (2) of the theorem.

In what follows we prove that $N_C(S_x)\subseteq N_C(x)$. Let $z\in N_C(S_x)$. In this case, it must exist a vertex $y\in S_x = S_1 - T$ such that $(y,z)\in E$. Since $y\notin T$, we must have $z\in T$, for otherwise the edge (y,z) would not be covered by T. Finally,

$$z\in T \quad\Rightarrow\quad z\in N_C(x) \quad \text{or} \quad z \in C-\{x\}.$$

Since $z\notin C$, we have that $z\in N_C(x)$. □

Theorem 2.2. *Let $G(V,E)$ be a simple graph and let C be a largest minimal cover of G. Then*

$$|V| \leqslant \frac{(|C|+2)^2}{4}.$$

Proof. Let $x\in C$ be the vertex with maximum external degree w.r.t. C. We divide the proof into two cases.

Case (i). There is a vertex $u\in C$ such that $N_C(x)\cap N_C(u)=\emptyset$.

For each vertex $v\in C$, let S_v be the set given by the previous lemma. Since $N_C(x)\cap N_C(u)=\emptyset$, it follows that $S_x\cap S_u=\emptyset$, for otherwise a vertex in the external neighborhood of $S_x\cap S_u$ w.r.t. C would belong to $N_C(x)\cap N_C(u)$. By using the same reasoning we can conclude that both $x \notin S_u \cup S_x$ and $u \notin S_u \cup S_x$. An upper bound on the size of V can be obtained as follows

$$|V| = |C| + \left|\bigcup_{v\in C} N_C(v)\right| = |C| + \left|\bigcup_{v\in C-S_x-S_u} N_C(v)\right|,$$

where the last equality holds because $N_C(S_x \cup S_u) \subseteq N_C(x) \cup N_C(u)$ and $x, u \in (C - S_x - S_u)$. By isolating u, we get that

$$|V| \leqslant |C| + \left|N_C(u)\right| + \left|\bigcup_{v\in C-S_x-S_u-\{u\}} N_C(v)\right| \leqslant |C| + \left|N_C(u)\right| + \sum_{v\in C-S_x-S_u-\{u\}} \left|N_C(v)\right|.$$

Since x is a vertex with the largest external degree, we have that

$$|V| \leqslant \big(|C| - |S_x| - |S_u| - 1\big) \times \left|N_C(x)\right| + \left|N_C(u)\right| + |C|.$$

Since $|S_x| \geqslant |N_C(x)| - 1$ we have that

$$|V| \leqslant \big(|C| - \left|N_C(x)\right| - |S_u|\big) \times \left|N_C(x)\right| + \left|N_C(u)\right| + |C|.$$

Finally, by using the fact that $|S_u| \geqslant |N_C(u)| - 1$, we conclude that the right-hand side of the expression above is only maximized w.r.t. S_u and $N_C(u)$, when $|S_u| = 0$ and $|N_C(u)| = 1$. Thus,

$$|V| \leqslant \big(|C| - N_C(x)\big) \times \left|N_C(x)\right| + 1 + |C|. \tag{1}$$

Case (ii). For every vertex $u \in C$, we have that $N_C(x) \cap N_C(u) \neq \emptyset$. In this case,

$$|V| = |C| + \left|\bigcup_{v\in C} N_C(v)\right| = |C| + \left|\bigcup_{v\in C-S_x} N_C(v)\right|.$$

Because x is a vertex with the largest external degree and $N_C(x) \cap N_C(u) \neq \emptyset$, for every $u \in C$, we have that

$$|V| \leqslant \big(|C| - |S_x|\big) \times \big(\left|N_C(x)\right| - 1\big) + 1 + |C|.$$

Since $|S_x| \geqslant |N_C(x)| - 1$, we have that

$$|V| \leqslant \big(|C| - \left|N_C(x)\right| + 1\big) \times \big(\left|N_C(x)\right| - 1\big) + 1 + |C|. \tag{2}$$

By maximizing the upper bounds given by Eqs. (1) and (2) as a real valued function in the variable $|N_C(x)|$, we establish our result. □

In order to show that our bound is tight we take K_{t+1}, a complete graph with $t + 1$ vertices, and to each vertex of K_{t+1} we connect t different vertices, each of them with degree 1. Note that $|V| = (t + 1) + (t + 1)t = (t + 1)^2$. In addition, the largest minimal cover C consists of t vertices from K_{t+1} and t vertices adjacent to the vertex of K_{t+1} that is not picked up. Thus, $|C| = 2t$ and $|V| = (|C| + 2)^2/4$.

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