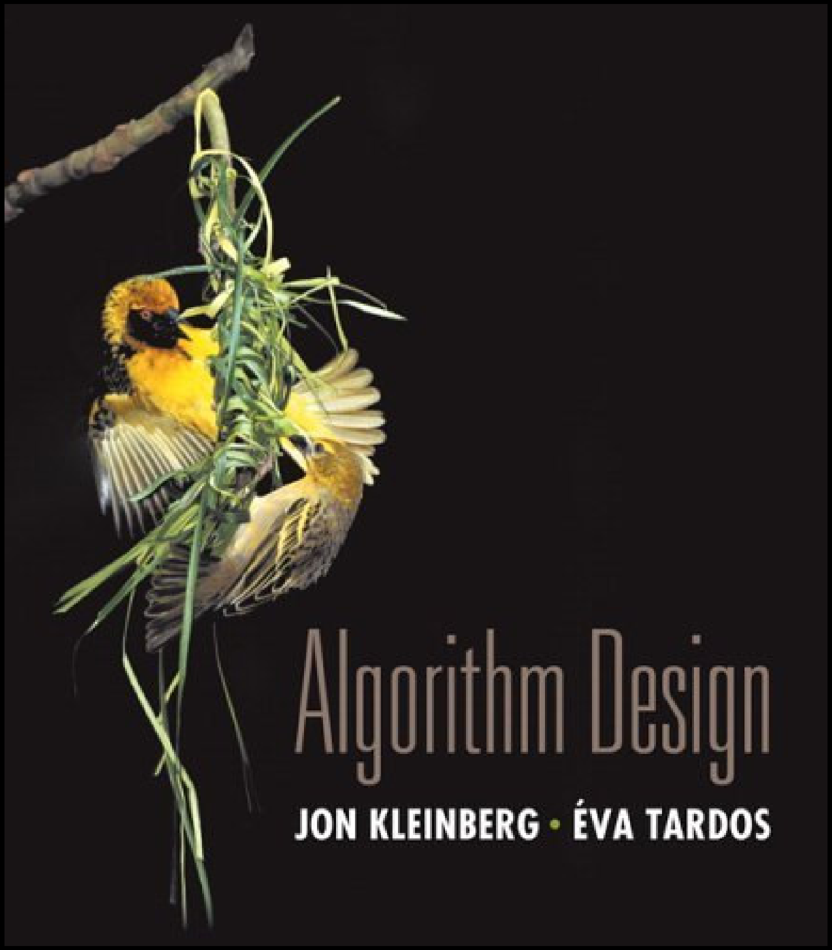


Chapter 13

Randomized Algorithms

Slides by Kevin Wayne.

Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization.

in practice, access to a pseudo-random number generator

Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Ex. Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

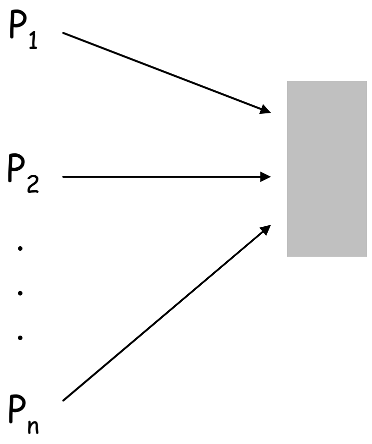
13.1 Contention Resolution

Contention Resolution in a Distributed System

Contention resolution. Given n processes $P_1, \ldots, P_n$, each competing for access to a shared database. If two or more processes access the database simultaneously, all processes are locked out. Devise protocol to ensure all processes get through on a regular basis.

Restriction. Processes can't communicate.

Challenge. Need symmetry-breaking paradigm.



Contention Resolution: Randomized Protocol

Protocol. Each process requests access to the database at time t with probability $p = 1/n$.

Claim. Let $S[i, t]$ = event that process i succeeds in accessing the database at time t. Then $1/(e \cdot n) \le \Pr[S(i, t)] \le 1/(2n)$.

Pf. By independence, $\Pr[S(i, t)] = p\,(1-p)^{n-1}$.

process i requests access

none of remaining n-1 processes request access

- Setting $p = 1/n$, we have $\Pr[S(i, t)] = 1/n\ \underbrace{(1 - 1/n)^{n-1}}_{\text{between } 1/e \text{ and } 1/2}$. ▪

value that maximizes $\Pr[S(i, t)]$

Useful facts from calculus. As n increases from 2, the function:

- $(1 - 1/n)^n$ converges monotonically from 1/4 up to 1/e
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to 1/e.

Contention Resolution: Randomized Protocol

Claim. The probability that process i fails to access the database in en rounds is at most 1/e. After $e \cdot n(c \ln n)$ rounds, the probability is at most n^{-c}.

Pf. Let F[i, t] = event that process i fails to access database in rounds 1 through t. By independence and previous claim, we have
$\Pr[F(i, t)] \leq (1 - 1/(en))^{t}$.

- Choose $t = \lceil e \cdot n \rceil$: $$\Pr[F(i,t)] \leq \left(1 - \tfrac{1}{en}\right)^{\lceil en \rceil} \leq \left(1 - \tfrac{1}{en}\right)^{en} \leq \tfrac{1}{e}$$
- Choose $t = \lceil e \cdot n \rceil \lceil c \ln n \rceil$: $$\Pr[F(i,t)] \leq \left(\tfrac{1}{e}\right)^{c \ln n} = n^{-c}$$

Contention Resolution: Randomized Protocol

Claim. The probability that all processes succeed within $2e \cdot n \ln n$ rounds is at least $1 - 1/n$.

Pf. Let $F[t]$ = event that at least one of the n processes fails to access database in any of the rounds 1 through t.

$$\Pr\big[F[t]\big] \;=\; \Pr\left[\bigcup_{i=1}^{n} F[i,t]\right] \;\underset{\substack{\uparrow \\ \text{union bound}}}{\le}\; \sum_{i=1}^{n} \Pr[F[i,t]] \;\underset{\substack{\uparrow \\ \text{previous slide}}}{\le}\; n\left(1-\tfrac{1}{en}\right)^{t}$$

- Choosing $t = 2\lceil en\rceil\lceil c \ln n\rceil$ yields $\Pr[F[t]] \le n \cdot n^{-2} = 1/n$. ▪

Union bound. Given events $E_1, \ldots, E_n$, $\Pr\left[\bigcup_{i=1}^{n} E_i\right] \le \sum_{i=1}^{n} \Pr[E_i]$

13.2 Global Minimum Cut

Global Minimum Cut

Global min cut. Given a connected, undirected graph G = (V, E) find a cut (A, B) of minimum cardinality.

Applications. Partitioning items in a database, identify clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

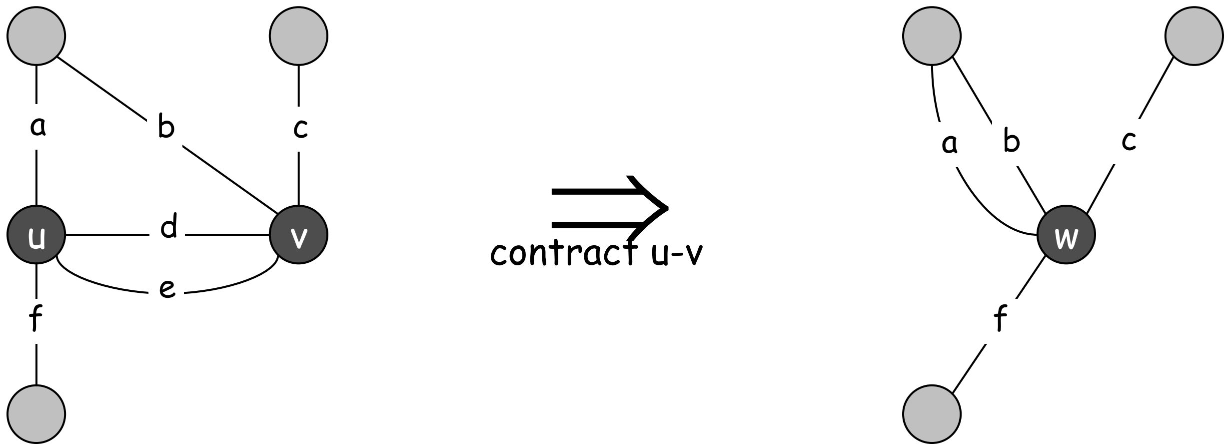
- Replace every edge (u, v) with two antiparallel edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

False intuition. Global min-cut is harder than min s-t cut.

Contraction Algorithm

Contraction algorithm. [Karger 1995]

- Pick an edge e = (u, v) uniformly at random.
- Contract edge e.
  - replace u and v by single new super-node w
  - preserve edges, updating endpoints of u and v to w
  - keep parallel edges, but delete self-loops
- Repeat until graph has just two nodes v_1 and v_2.
- Return the cut (all nodes that were contracted to form v_1).



Probabilidade

Lema 3: Seja t_1, t_2, ..., t_k uma coleção de eventos. Temos que

$$\Pr\left[\bigcap_{i=1}^{k} t_i\right] = \prod_{i=1}^{k} \Pr\left[t_i \,\middle|\, \bigcap_{j=1}^{i-1} t_j\right]$$

Prova:

Base:

$$\Pr[t_2|t_i] = \frac{\Pr[t_2 \cap t_1]}{\Pr[t_1]} \quad \Rightarrow \quad \Pr(t_1 \cap t_2) = \Pr(t_1)\cdot\Pr(t_2|t_1) \quad \text{ok}$$

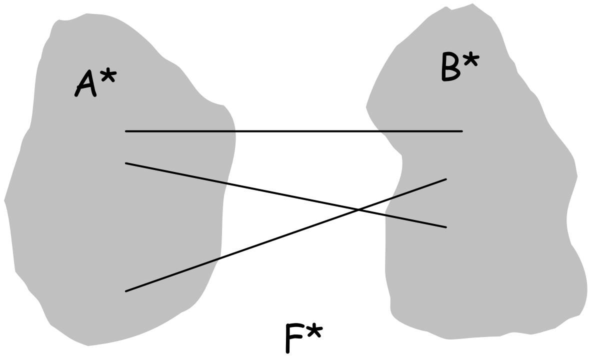
Assuma que vale para k, provar para k+1.

Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob $\geq 2/n^2$.

Pf. Consider a global min-cut (A^*, B^*) of G. Let F^* be edges with one endpoint in A^* and the other in B^*. Let $k = |F^*|$ = size of min cut.

- In first step, algorithm contracts an edge in F^* probability $k / |E|$.
- Every node has degree $\geq k$ since otherwise (A^*, B^*) would not be min-cut. $\Rightarrow$ $|E| \geq \frac{1}{2}kn$.
- Thus, algorithm contracts an edge in F^* with probability $\leq 2/n$.



Contraction Algorithm

Claim. The contraction algorithm returns a min cut with prob ≥ $2/n^2$.

Pf. Consider a global min-cut (A*, B*) of G. Let F* be edges with one endpoint in A* and the other in B*. Let k = |F*| = size of min cut.

- Let G' be graph after j iterations. There are n' = n-j supernodes.
- Suppose no edge in F* has been contracted. The min-cut in G' is still k.
- Since value of min-cut is k, |E'| ≥ $\frac{1}{2}$kn'.
- Thus, algorithm contracts an edge in F* with probability ≤ 2/n'.

- Let E_j = event that an edge in F* is not contracted in iteration j.

$$
\begin{aligned}
\Pr[E_1 \cap E_2 \cdots \cap E_{n-2}] &= \Pr[E_1] \times \Pr[E_2 \mid E_1] \times \cdots \times \Pr[E_{n-2} \mid E_1 \cap E_2 \cdots \cap E_{n-3}] \\
&\geq \left(1-\tfrac{2}{n}\right)\left(1-\tfrac{2}{n-1}\right)\cdots\left(1-\tfrac{2}{4}\right)\left(1-\tfrac{2}{3}\right) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\cdots\left(\tfrac{2}{4}\right)\left(\tfrac{1}{3}\right) \\
&= \frac{2}{n(n-1)} \\
&\geq \frac{2}{n^2}
\end{aligned}
$$

Contraction Algorithm

Amplification. To amplify the probability of success, run the contraction algorithm many times and return the best solution

Claim. If we repeat the contraction algorithm $n^2 \ln n$ times with independent random choices, the probability of failing to find the global min-cut is at most $1/n^2$.

Pf. By independence, the probability of failure is at most

$$\left(1-\frac{2}{n^2}\right)^{n^2\ln n} = \left[\left(1-\frac{2}{n^2}\right)^{\frac{1}{2}n^2}\right]^{2\ln n} \underset{\substack{\uparrow \\ (1 - 1/x)^x \le 1/e}}{\le} \left(e^{-1}\right)^{2\ln n} = \frac{1}{n^2}$$

Global Min Cut: Context

Remark. Overall running time is slow since we perform $\Theta(n^2 \log n)$ iterations and each takes $\Omega(m)$ time.

Improvement. [Karger-Stein 1996] $O(n^2 \log^3 n)$.

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm until $n / \sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Extensions. Naturally generalizes to handle positive weights.

Best known. [Karger 2000] $O(m \log^3 n)$.

faster than best known max flow algorithm or deterministic global min cut algorithm

13.3 Linearity of Expectation

Expectation

Expectation. Given a discrete random variables X, its expectation E[X] is defined by:

$$E[X] = \sum_{j=0}^{\infty} j \Pr[X = j]$$

Waiting for a first success. Coin is head with probability p and tails with probability 1-p. How many independent flips X until first heads?

$$E[X] \;=\; \sum_{j=0}^{\infty} j \cdot \Pr[X = j] \;=\; \sum_{j=0}^{\infty} j \,\underset{\substack{\uparrow \\ \text{j-1 tails}}}{(1-p)^{j-1}}\, \underset{\substack{\uparrow \\ \text{1 head}}}{p} \;=\; \frac{p}{1-p} \sum_{j=0}^{\infty} j\,(1-p)^{j} \;=\; \frac{p}{1-p} \cdot \frac{1-p}{p^2} \;=\; \frac{1}{p}$$

Expectation: Two Properties

Useful property. If X is a 0/1 random variable, E[X] = Pr[X = 1].

Pf.

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X=j] = \sum_{j=0}^{1} j \cdot \Pr[X=j] = \Pr[X=1]$$

not necessarily independent

Linearity of expectation. Given two random variables X and Y defined over the same probability space, E[X + Y] = E[X] + E[Y].

Decouples a complex calculation into simpler pieces.

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Memoryless guessing. No psychic abilities; can't even remember what's been turned over already. Guess a card from full deck uniformly at random.

Claim. The expected number of correct guesses is 1.

Pf. (surprisingly effortless using linearity of expectation)

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1/n$.
- $E[X] \underset{\substack{\uparrow \\ \text{linearity of expectation}}}{=} E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/n = 1.$ ▪

Guessing Cards

Game. Shuffle a deck of n cards; turn them over one at a time; try to guess each card.

Guessing with memory. Guess a card uniformly at random from cards not yet seen.

Claim. The expected number of correct guesses is $\Theta(\log n)$.

Pf.

- Let X_i = 1 if i^{th} prediction is correct and 0 otherwise.
- Let X = number of correct guesses = $X_1 + \ldots + X_n$.
- $E[X_i] = \Pr[X_i = 1] = 1 / (n - i - 1)$.
- $E[X] \underset{\substack{\uparrow \\ \text{linearity of expectation}}}{=} E[X_1] + \ldots + E[X_n] = 1/n + \ldots + 1/2 + 1/1 \underset{\substack{\uparrow \\ \ln(n+1) < H(n) < 1 + \ln n}}{=} H(n)$. ▪

Coupon Collector

Coupon collector. Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, how many boxes before you have ≥ 1 coupon of each type?

Claim. The expected number of steps is Θ(n log n).

Pf.

- Phase j = time between j and j+1 distinct coupons.
- Let X_j = number of steps you spend in phase j.
- Let X = number of steps in total = $X_0 + X_1 + \ldots + X_{n-1}$.

$$E[X] = \sum_{j=0}^{n-1} E[X_j] \underset{\uparrow}{=} \sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{i=1}^{n} \frac{1}{i} = n\,H(n)$$

prob of success = (n-j)/n
⇒ expected waiting time = n/(n-j)

13.5 Randomized Divide-and-Conquer

Quicksort

Sorting. Given a set of n distinct elements S, rearrange them in ascending order.

```
RandomizedQuicksort(S) {
   if |S| = 0 return

   choose a splitter $a_i$ ∈ S uniformly at random
   foreach (a ∈ S) {
      if      (a < $a_i$) put a in $S^-$
      else if (a > $a_i$) put a in $S^+$
   }
   RandomizedQuicksort($S^-$)
   output $a_i$
   RandomizedQuicksort($S^+$)
}
```

Remark. Can implement in-place.

↑ O(log n) extra space

Quicksort

Running time.

- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

Randomize. Protect against worst case by choosing splitter at random.

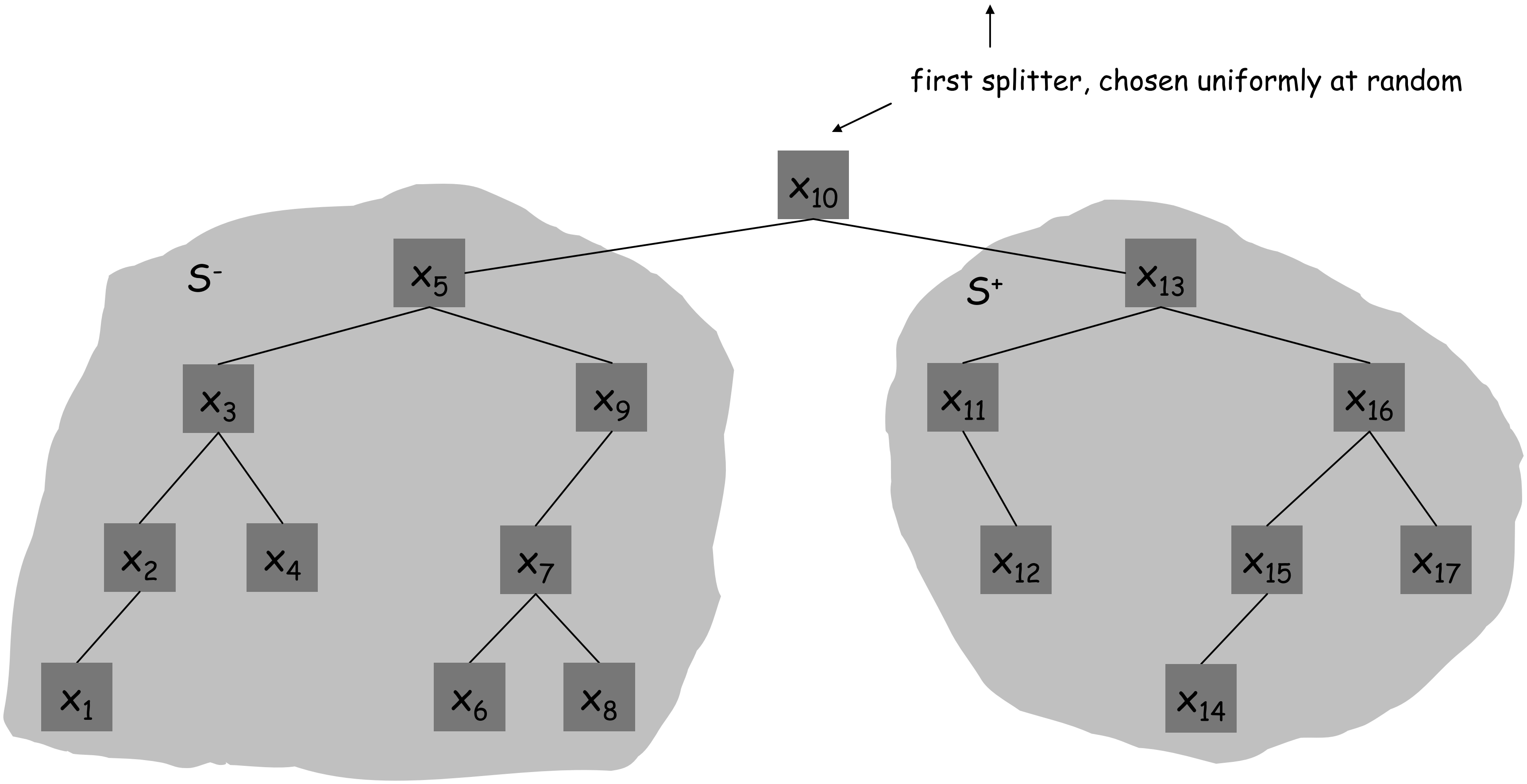
Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

Quicksort: BST Representation of Splitters

BST representation. Draw recursive BST of splitters.

x_7	x_6	x_{12}	x_3	x_{11}	x_8	x_7	x_1	x_{15}	x_{13}	x_{17}	x_{10}	x_{16}	x_{14}	x_9	x_4	x_5



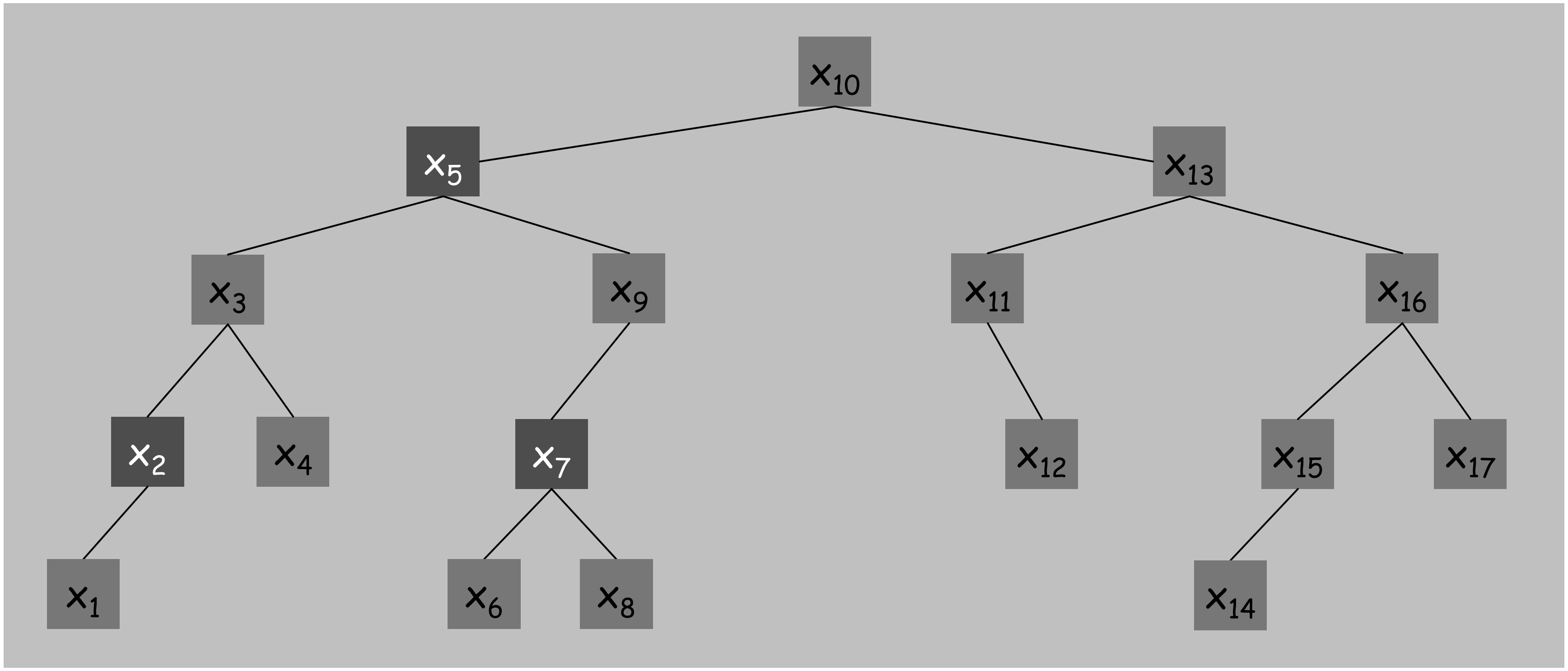
first splitter, chosen uniformly at random

Quicksort: BST Representation of Splitters

Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7.
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6.

Claim. $\Pr[x_i \text{ and } x_j \text{ are compared}] = 2 / |j - i + 1|$.



Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is O(n log n).

Pf.

$$\sum_{1 \le i < j \le n} \underset{\substack{\uparrow \\ \text{probability that i and j are compared}}}{\frac{2}{j-i+1}} = 2 \sum_{i=1}^{n} \sum_{j=2}^{i} \frac{1}{j} \le 2n \sum_{j=1}^{n} \frac{1}{j} \approx 2n \int_{x=1}^{n} \frac{1}{x} dx = 2n \ln n$$

Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is O(n log n).

Pf.

$$\sum_{1 \le i < j \le n} \underset{\substack{\uparrow \\ \text{probability that i and j are compared}}}{\frac{2}{j-i+1}} \;=\; 2\sum_{i=1}^{n}\sum_{j=2}^{i}\frac{1}{j} \;\le\; 2n\sum_{j=1}^{n}\frac{1}{j} \;\approx\; 2n\int_{x=1}^{n}\frac{1}{x}\,dx \;=\; 2n\ln n$$

Theorem. [Knuth 1973] Stddev of number of comparisons is ~ 0.65N.

Ex. If n = 1 million, the probability that randomized quicksort takes less than 4n ln n comparisons is at least 99.94%.

> **Chebyshev's inequality.** $\Pr[|X - \mu| \ge k\delta] \le 1 / k^2$.

13.4 MAX 3-SAT

Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$\begin{array}{lcl}
C_1 & = & x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 & = & x_2 \vee x_3 \vee \overline{x_4} \\
C_3 & = & \overline{x_1} \vee x_2 \vee x_4 \\
C_4 & = & \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 & = & x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{array}$$

Remark. NP-hard search problem.

Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$\begin{array}{lcl} C_1 & = & x_2 \vee \overline{x_3} \vee \overline{x_4} \\ C_2 & = & x_2 \vee x_3 \vee \overline{x_4} \\ C_3 & = & \overline{x_1} \vee x_2 \vee x_4 \\ C_4 & = & \overline{x_1} \vee \overline{x_2} \vee x_3 \\ C_5 & = & x_1 \vee \overline{x_2} \vee \overline{x_4} \end{array}$$

Remark. NP-hard search problem.

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Maximum 3-Satisfiability: Analysis

Claim. Given a 3-SAT formula with k clauses, the expected number of clauses satisfied by a random assignment is 7k/8.

Pf. Consider random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$

- Let Z = number of clauses satisfied by the assignment.

$$\begin{aligned} E[Z] &= \sum_{j=1}^{k} E[Z_j] \\ \text{linearity of expectation} \nearrow \quad &= \sum_{j=1}^{k} \Pr[\text{clause } C_j \text{ is satisfied}] \\ &= \tfrac{7}{8} k \end{aligned}$$

The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Probabilistic method. We showed the existence of a non-obvious property of 3-SAT by showing that a random construction produces it with positive probability!

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies ≥ 7k/8 clauses is at least 1/(8k).

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that ≥ 7k/8 clauses are satisfied.

$$
\begin{aligned}
\tfrac{7}{8}k \;=\; E[Z] &= \sum_{j \ge 0} j\,p_j \\
&= \sum_{j < 7k/8} j\,p_j \;+\; \sum_{j \ge 7k/8} j\,p_j \\
&\le \left(\tfrac{7k}{8} - \tfrac{1}{8}\right) \sum_{j < 7k/8} p_j \;+\; k \sum_{j \ge 7k/8} p_j \\
&\le \left(\tfrac{7}{8}k - \tfrac{1}{8}\right) \cdot 1 \;+\; k\,p
\end{aligned}
$$

Rearranging terms yields p ≥ 1 / (8k). ▪

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies ≥ 7k/8 clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least 1/(8k). By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most 8k. ▪

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

very unlikely to improve over simple randomized
algorithm for MAX-3SAT

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed to run in poly-time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas algorithm. Guaranteed to find correct answer, likely to run in poly-time.
Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

stop algorithm after a certain point
↓

Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with one-sided error in poly-time.

Can decrease probability of false negative to 2^{-100} by 100 independent repetitions

One-sided error.

- If the correct answer is `no`, always return `no`.
- If the correct answer is `yes`, return `yes` with probability $\geq \frac{1}{2}$.

ZPP. [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but on average it is fast

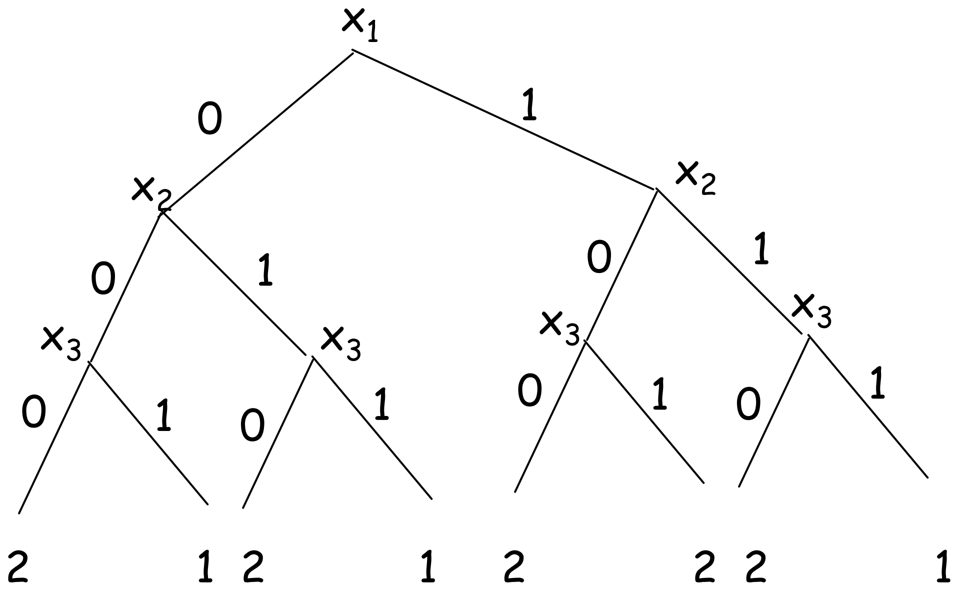
Theorem. $P \subseteq ZPP \subseteq RP \subseteq NP$.

Fundamental open questions. To what extent does randomization help? Does P = ZPP? Does ZPP = RP? Does RP = NP?

MAX SAT: Desaleatorização

$C_1 = (x_1 \vee x_2 \vee \neg x_3)$

$C_2 = (\neg x_2 \vee \neg x_3)$



- Cada folha da árvore corresponde a um atribuição:
- Cada folha esta associada ao número de clausulas satisfeitas pela atribuição correspondente

MAX SAT: Desaleatorização

Métodos das Probabilidades Condicionais

$$E[X] = 1/2.E[X|x_1=1]+1/2.E[X|x_1=0] = 7/8\ k$$

$$\Downarrow$$

(i) $E[X|x_1=1] >=7/8k$ ou

(ii) $E[X|x_1=0] >=7/8k$

Se (i) fixamos $x_1=1$, caso contrário fixamos $x_1=0$.

Consideramos x_2 recursivamente e decemos até chegar uma folha

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

`Create()`:	Initialize a dictionary with $S = \phi$.
`Insert(u)`:	Add element $u \in U$ to S.
`Delete(u)`:	Delete u from S, if u is currently in S.
`Lookup(u)`:	Determine whether u is in S.

Challenge. Universe U can be extremely large so defining an array of size |U| is infeasible.

Applications. File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc.

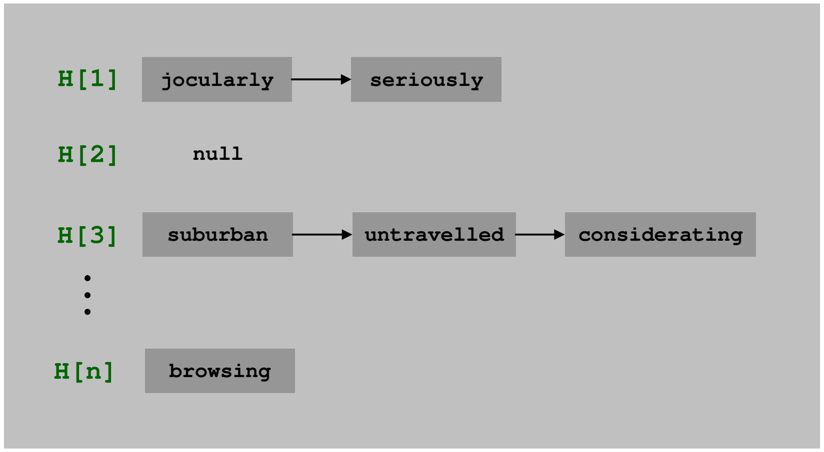
Hashing

Hash function. $h : U \rightarrow \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array H of size n. When processing element u, access array element H[h(u)].

Collision. When $h(u) = h(v)$ but $u \neq v$.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions. This phenomenon is known as the "birthday paradox."
- Separate chaining: H[i] stores linked list of elements u with $h(u) = i$.



Ad Hoc Hash Function

Ad hoc hash function.

```
int h(String s, int n) {
   int hash = 0;
   for (int i = 0; i < s.length(); i++)
      hash = (31 * hash) + s[i];
   return hash % n;
}
```

hash function ala Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

Q. But isn't ad hoc hash function good enough in practice?

Algorithmic Complexity Attacks

When can't we live with ad hoc hash function?

- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary learns your ad hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = α = m / n.
- Choose n $\approx$ m $\Rightarrow$ on average O(1) per insert, lookup, or delete.

Challenge. Achieve idealized randomized guarantees, but with a hash function where you can easily find items where you put them.

Approach. Use randomization in the choice of h.

adversary knows the randomized algorithm you're using,
but doesn't know random choices that the algorithm makes

Universal Hashing

Universal class of hash functions. [Carter-Wegman 1980s]

- For any pair of elements u, v ∈ U, $\Pr_{h \in H}\left[h(u) = h(v) \right] \le 1/n$
- Can select random h efficiently.
- Can compute h(u) efficiently.

chosen uniformly at random

Ex. U = { a, b, c, d, e, f }, n = 2.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$
$\Pr_{h \in H}[h(a) = h(b)] = 1/2$
$\Pr_{h \in H}[h(a) = h(c)] = 1$
$\Pr_{h \in H}[h(a) = h(d)] = 0$
. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$
$\Pr_{h \in H}[h(a) = h(b)] = 1/2$
$\Pr_{h \in H}[h(a) = h(c)] = 1/2$
$\Pr_{h \in H}[h(a) = h(d)] = 1/2$
$\Pr_{h \in H}[h(a) = h(e)] = 1/2$
$\Pr_{h \in H}[h(a) = h(f)] = 0$
. . .

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let h $\in$ H be chosen uniformly at random from H; and let u $\in$ U. For any subset S $\subseteq$ U of size at most n, the expected number of items in S that collide with u is at most 1.

Pf. For any element s $\in$ S, define indicator random variable X_s = 1 if h(s) = h(u) and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h\in H}[X] \;=\; E[\textstyle\sum_{s\in S} X_s] \;\underset{\substack{\uparrow \\ \text{linearity of expectation}}}{=}\; \sum_{s\in S} E[X_s] \;\underset{\substack{\uparrow \\ X_s \text{ is a 0-1 random variable}}}{=}\; \sum_{s\in S} \Pr[X_s = 1] \;\underset{\substack{\uparrow \\ \text{universal} \\ \text{(assumes } u \notin S)}}{\le}\; \sum_{s\in S} \tfrac{1}{n} \;=\; |S|\,\tfrac{1}{n} \;\le\; 1$$

Designing a Universal Family of Hash Functions

Theorem. [Chebyshev 1850] There exists a prime between n and 2n.

Modulus. Choose a prime number $p \approx n$. ← no need for randomness here

Integer encoding. Identify each element $u \in U$ with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \le a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a Universal Class of Hash Functions

Theorem. $H = \{ h_a : a \in A \}$ is a universal class of hash functions.

Pf. Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ be two distinct elements of U. We need to show that $\Pr[h_a(x) = h_a(y)] \le 1/n$.

- Since $x \ne y$, there exists an integer j such that $x_j \ne y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} = \underbrace{\sum_{i \ne j} a_i (x_i - y_i)}_{m} \mod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \ne j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \ne j$.
- Since p is prime, $a_j z = m \bmod p$ has at most one solution among p possibilities. ← see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] = 1/p \le 1/n$. ▪

Number Theory Facts

Fact. Let p be prime, and let $z \neq 0 \bmod p$. Then $\alpha z = m \bmod p$ has at most one solution $0 \leq \alpha < p$.

Pf.

- Suppose α and β are two different solutions.
- Then $(\alpha - \beta)z = 0 \bmod p$; hence $(\alpha - \beta)z$ is divisible by p.
- Since $z \neq 0 \bmod p$, we know that z is not divisible by p; it follows that $(\alpha - \beta)$ is divisible by p.
- This implies $\alpha = \beta$. ▪

Bonus fact. Can replace "at most one" with "exactly one" in above fact.
Pf idea. Euclid's algorithm.

13.9 Chernoff Bounds

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

sum of independent 0-1 random variables
is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any $t > 0$,

$$\Pr[X > (1+\delta)\mu] = \Pr\left[e^{tX} > e^{t(1+\delta)\mu}\right] \leq e^{-t(1+\delta)\mu} \cdot E[e^{tX}]$$

$f(x) = e^{tX}$ is monotone in x

Markov's inequality: $\Pr[X > a] \leq E[X] / a$

- Now $E[e^{tX}] = E[e^{t \sum_i X_i}] = \prod_i E[e^{t X_i}]$

definition of X

independence

Chernoff Bounds (above mean)

Pf. (cont)

- Let $p_i = \Pr[X_i = 1]$. Then,

$$E[e^{tX_i}] \;=\; p_i e^t + (1-p_i)e^0 \;=\; 1 + p_i(e^t - 1) \;\underset{\substack{\uparrow \\ \text{for any } \alpha \geq 0,\ 1+\alpha \leq e^{\alpha}}}{\leq}\; e^{p_i(e^t - 1)}$$

- Combining everything:

$$\Pr[X > (1+\delta)\mu] \;\underset{\substack{\uparrow \\ \text{previous slide}}}{\leq}\; e^{-t(1+\delta)\mu} \prod_i E[e^{tX_i}] \;\underset{\substack{\uparrow \\ \text{inequality above}}}{\leq}\; e^{-t(1+\delta)\mu} \prod_i e^{p_i(e^t-1)} \;\underset{\substack{\uparrow \\ \sum_i p_i = E[X] \;\leq\; \mu}}{\leq}\; e^{-t(1+\delta)\mu}\; e^{\mu(e^t-1)}$$

- Finally, choose $t = \ln(1 + \delta)$. ▪

Chernoff Bounds (below mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \le E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] \; < \; e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load Balancing

Analysis.

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \frac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound ⇒ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n / \log\log n)$ jobs.

Fact: this bound is asymptotically tight: with high probability, some processor receives $\Theta(\log n / \log\log n)$

Load Balancing: Many Jobs

Theorem. Suppose the number of jobs m = 16n ln n. Then on average, each of the n processors handles μ = 16 ln n jobs. With high probability every processor will have between half and twice the average load.

Pf.

- Let X_i , Y_{ij} be as before.
- Applying Chernoff bounds with δ = 1 yields

$$\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16n \ln n} < \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n^2} \qquad \Pr[X_i < \tfrac{1}{2}\mu] < e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2 (16n \ln n)} = \frac{1}{n^2}$$

- Union bound ⇒ every processor has load between half and twice the average with probability ≥ 1 - 2/n. ▪