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Contributions to a Proof Theory for Generic Defaults

Ramiro A. de T. Guerreiro, Marco A. Casanova and Andrea S. Hemerly

Rio Scientific Center - IBM Brazil
P.O. Box 4624
20.001, Rio de Janeiro, RJ - Brazil

ABSTRACT

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1. INTRODUCTION AND MOTIVATION

Default logic (Reiter [1980]) provides an interesting approach to formalize nonmonotonic reasoning. Informally, a default theory Δ consists of a set of sentences W and a set of defaults D, where a default is an expression of the form $(\alpha:\beta_1,\ldots,\beta_m/\omega)$, whose meaning is "if α holds and it is consistent to assume $\beta_1,\ldots,\beta_m$ then infer ω". The default theory Δ defines a set of first-order theories, its *extensions*, obtained from W by firing defaults in a consistent way. A sentence α is considered to be a consequence of Δ iff α is in one of the extensions of Δ.

However, the development of default logic presents a gap since Reiter describes a proof theory only for normal default theories, that is, those theories that admit only defaults of the form $(\alpha:\omega/\omega)$. The reason that the proof theory does not apply to arbitrary default theories is intimately connected with his notion of extension.

The first contribution of this paper is to close this gap by defining a proof theory for arbitrary default theories. The proof theory is shown to be sound and complete with respect to C-extensions, an alternative to Reiter's notion of extension.

Essentially, C-extensions allow defaults to be ignored, but no default is dispensed of unless it is strictly necessary. C-extensions will satisfy a minimality property and a semi-monotonicity property. Moreover, every default theory will have a C-extension. Finally, the notion of C-extension will coincide with Reiter's notion of extension for normal default theories.

Etherington [1987] studies semi-normal default theories, that is, theories that admit only defaults of the form $(\alpha:\omega\wedge\beta/\omega)$. He establishes conditions under which a semi-normal default theory has an extension (in Reiter's sense) but he does not address proof theoretic questions. Another notion of extension can be found in Lukaszewicz [1984].

Naturally, there remains the question of obtaining alternatives to the notion of C-extension with similar properties. The second contribution of this paper is a partial answer to this question. It provides necessary conditions that a notion of extension must satisfy in order to permit defining a sound and complete proof theory for arbitrary default theories. The results in this case are based on abstract definitions for the concepts of extension and default proof that extend those introduced in Reiter [1980] and in the first half of the paper.

The organization of the paper is as follows. Section 2 reviews the basic concepts and results of Reiter's default logic. Section 3 introduces the concept of C-extension and section 4, a proof theory which is sound and complete with respect to this concept. Section 5 considers the more abstract question of obtaining conditions on alternative definitions for the concept of extension that lead to adequate proof theories. Finally, section 6 contains the conclusions.

2. A BRIEF REVIEW OF REITER'S DEFAULT LOGIC

In this section, we briefly review some concepts of default logic, as defined in Reiter [1980].

A *default* is an expression of the form $(\alpha:\beta_1,...,\beta_m/\omega)$ where α, $\beta_1,...,\beta_m$ and ω are all first-order formulas. The formulas α and ω are called, respectively, the *prerequisite* and the *consequent* of the default, whereas the formulas $\beta_1,...,\beta_m$ are called the *justifications*. A default is *closed* iff $\alpha,\beta_1,...,\beta_m$ and ω are sentences, that is, first-order formulas with no free variables. Otherwise, the default is *open*. A *normal default* is a default of the form $(\alpha:\omega/\omega)$.

A *default theory* is a pair $\Delta=(D,W)$, where D is a set of defaults and W is a set of first-order sentences. A default theory is *open, closed or normal* iff D is a set of open, closed or normal defaults. As we will deal only with closed defaults we will omit the word "closed" from now on (the reader should bear this in mind when reading the paper).

Given a set of defaults D, we define:

$$\mathsf{CONSEQ}(D)=\{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega}\in D\}$$

$$\mathsf{PREREQ}(D)=\{\alpha \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega}\in D\}$$

$$\mathsf{JUSTIF}(D)=\{\beta_i \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega}\in D \text{ and } i\in[1,m]\}$$

For any set of sentences S and any formula ω, we use $S\vdash\omega$ to indicate that ω is a theorem of S (in some specific calculus for classic first-order logic). We also define $\mathsf{Th}(S)$ as the set of first-order sentences ω such that $S\vdash\omega$.

A default theory is associated with a set of first-order theories, its *R-extensions* (we prefer to call R-extension the concept of extension introduced in Reiter [1980] to emphasize that it is just one of the alternatives we will consider in this paper). The following theorem provides a characterization of R-extensions.

Theorem 1:

Let E be a set of sentences and let $\Delta=(D,W)$ be a default theory. Define:

$$E_0=W$$

and, for $i\geq 0$:

$$E_{i+1}=\mathsf{Th}(E_i)\ \cup\ \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega}\in D, \text{ where } \alpha\in E_i \text{ and } \neg\beta_1,\ldots,\neg\beta_m\notin E\}$$

Then, E is a R-extension for Δ iff $E=\bigcup_{i=0}^{\infty} E_i$.

We then say that a sentence β is a *R-consequence* of a default theory Δ iff there is a R-extension E of Δ such that $\beta\in E$.

Given a default theory $\Delta=(D,W)$ and a R-extension E of Δ, the set of *generating defaults* for E with respect to Δ is defined as:

$$\mathsf{RGD}(E,\Delta)=\{\frac{\alpha:\beta_1,\ldots,\beta_m}{\omega}\in D \mid \alpha\in E \text{ and } \neg\beta_1,\ldots,\neg\beta_m\notin E\}$$

Unlike all other results in this section, the first property of R-Extensions we state is not directly taken from Reiter [1980].

Theorem 2: *(Strong minimality of R-extensions)*

Let $\Delta=(D,W)$ and $\Delta'=(D',W')$ be default theories such that $D'\subseteq D$ and $W'\subseteq W$. For any R-extension E for Δ and any R-extension E' for Δ', if $E\subseteq E'$ then $E=E'$.

Proof

Let $\Delta=(D,W)$ and $\Delta'=(D',W')$ be two default theories such that $D'\subseteq D$ and $W'\subseteq W$. Let E be a R-extension for Δ and E' be a R-extension for Δ' and suppose that $E\subseteq E'$. By definition, $E=\bigcup_{i=0}^{\infty} E_i$ and $E'=\bigcup_{i=0}^{\infty} F_i$, where

(1a) $E_0=W$

(1b) $F_0=W'$

and, for $i\geq 0$:

(2a) $E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D \text{ where } \alpha\in E_i,\ \neg\beta_1,\ldots,\neg\beta_m\notin E\}$

(2b) $F_{i+1} = Th(F_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D' \text{ where } \alpha\in F_i,\ \neg\beta_1,\ldots,\neg\beta_m\notin E'\}$

Since $E\subseteq E'$, we only have to prove that $E'\subseteq E$. This will be achieved by showing that $F_j\subseteq E_j$, for all $j\geq 0$. For $j=0$ we trivially have $F_0\subseteq E_0$, since $W'\subseteq W$ by assumption. Let $i\geq 0$ and assume that $F_i\subseteq E_i$. We will prove that $F_{i+1}\subseteq E_{i+1}$. Let $\omega\in F_{i+1}$. If $\omega\in Th(F_i)$, then $\omega\in E_{i+1}$ since $F_i\subseteq E_i$, by assumption, and $Th(E_i)\subseteq E_{i+1}$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega)\in D'$ such that $\alpha\in F_i$ and $\neg\beta_1,\ldots,\neg\beta_m\notin E'$. As $F_i\subseteq E_i$, α is also in E_i. Since $D'\subseteq D$ and $E\subseteq E'$, $(\alpha:\beta_1,\ldots,\beta_m/\omega)\in D$ and $\neg\beta_1,\ldots,\neg\beta_m\notin E$. Then, by (2a), $\omega\in E_{i+1}$. Thus $F_{i+1}\subseteq E_{i+1}$. Hence $E'=\bigcup_{i=0}^{\infty} F_i \subseteq \bigcup_{i=0}^{\infty} E_i = E$.

□

Corollary 1: *(Minimality of R-extensions)*

Let $\Delta=(D,W)$ be a default theory. For any two R-extensions E and F for Δ, if $E\subseteq F$ then $E=F$.

In addition, for normal default theories only, we also have:

Theorem 3: *(Existence of R-extensions)*

Every normal default theory has a R-extension.

Theorem 4: *(Semi-Monotonicity of R-extensions)*

Let $\Delta=(D,W)$ and $\Delta'=(D',W')$ be two normal default theories such that $D'\subseteq D$ and $W'=W$. For any R-extension E' for Δ', there is a R-extension E for Δ such that $E'\subseteq E$.

Again only for normal default theories, Reiter [1980] introduces a proof theory which is sound and complete with respect to R-extensions.

Definition 1:

Let $\Delta=(D,W)$ be a normal default theory and γ be a sentence. A finite sequence $<D_0,...,D_k>$ of finite subsets of D is a *R-default proof* of γ from Δ iff:

P_1. $W \cup \mathrm{CONSEQ}(D_0) \vdash \gamma$

P_2. For $1 \le i \le k$, $W \cup \mathrm{CONSEQ}(D_i) \vdash \mathrm{PREREQ}(D_{i-1})$

P_3. $D_k=\emptyset$

P_4. If W is consistent, then $W \cup \bigcup_{i=0}^{k} \mathrm{CONSEQ}(D_i)$ must be consistent.

The proviso in P_4 that W be consistent is just a technical convenience and it implies that any finite sequence $<D_0,...,D_k>$ of finite subsets of D such that $D_k=0$ will be a R-default proof of any sentence α from $\Delta=(D,W)$, if W is inconsistent. It is not part of Reiter's original definition and, as a consequence, his completeness result holds only for normal default theories whose set of sentences is consistent.

The last result that we will highlight concerns the soundness and completeness of R-default proofs.

Theorem 5: *(Soundness and Completeness of R-Default Proofs)*

Let $\Delta=(D,W)$ be a normal default theory. Then, Δ has a R-extension which contains the sentence β iff there is a R-default proof of β from Δ.

The above results can be extended to open normal default theories. However, we emphasize that, except for the minimality of R-extensions, they do not hold for arbitrary (closed or open) default theories.

3. C-EXTENSIONS

In this section we introduce the concept of C-extension and prove its basic properties. To motivate C-extensions, consider the following situation (the Barber's Paradox):

"Beardland is a small city where the barber Noel shaves every citizen who does not shave himself. Does Noel shave himself ?"

To formalize the Barber's Paradox, we may try to define a default theory Δ consisting of an empty set of sentences and the set of all defaults of the form:

$$\frac{:\neg \text{shaves}(t, t)}{\text{shaves}(\text{Noel}, t)}$$

where t is any closed term of the language in question.

However, this formalization fails because Δ has no R-extensions since the default $D_1 = (:\neg\,\text{shaves(Noel,Noel)/shaves(Noel,Noel)})$ is present in the theory. As a consequence, we cannot prove that the barber Noel shaves Golias, even assuming that Golias is a citizen of Beardland that does not shave himself. To conclude this, we would have to disregard D_1, something that the definition of R-extension does not allow. More precisely, suppose that Golias is a constant in our alphabet. Even though it is consistent to assume $\neg\,\text{shaves(Golias,Golias)}$, which would sanction the use of the default $D_2 = (:\neg\,\text{shaves(Golias,Golias)/shaves(Noel,Golias)})$, we cannot conclude that shaves(Noel,Golias) is a R-consequence of Δ because Δ has no R-extensions.

C-extensions provide a solution to this problem since they allow some defaults to be ignored. Hence, Δ will have a C-extension containing the fact shaves(Noel,Golias), generated using the default $D_2 = (:\neg\,\text{shaves(Golias,Golias)/shaves(Noel,Golias)})$, but ignoring the default $D_1 = (:\neg\,\text{shaves(Noel,Noel)/shaves(Noel,Noel)})$.

However, the power of ignoring defaults may lead to a large number of extensions, if an agent gives them up arbitrarily. In our approach, this problem is avoided because C-extensions are defined as maximal sets, i.e., no default is dispensed of unless it is strictly necessary. Thus, C-extensions will satisfy a minimality property. Moreover, every default theory will have a C-extension, since the defaults which would forbid a default theory to have a C-extension may be ignored. Furthermore, every R-extension is a C-extension and, in fact, the two concepts coincide for normal default theories. Therefore, the differences between the two concepts appear only when we consider arbitrary default theories.

To define C-extensions, we need the notion of E-sets:

Definition 2:

Let $\Delta = (D, W)$ be a default theory. A set E of first-order sentences is an *E-set* for Δ iff $E = \bigcup_{i=1}^{\infty} E_i$, where:

$$E_0 = W$$

and for $i \geq 0$:

$$E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D \text{ where } \alpha \in E_i,\ \neg\beta_1,\ldots,\neg\beta_m \notin E \text{ and } \omega \in E\}$$

Note that this definition is quite similar to the statement of Theorem 1, except that we require that $\omega \in E$ in the definition of E_{i+1}.

Let $\Delta = (D, W)$ be a default theory. The set of *generating defaults* for an E-set E of Δ is defined as:

$$EGD(E,\Delta) = \{ \frac{\alpha : \beta_1, \ldots, \beta_m}{\omega} \in D \mid \alpha \in E,\ \neg\beta_1,\ldots,\neg\beta_m \notin E \text{ and } \omega \in E\}$$

We also introduce the following sets:

$$E(\Delta) = \{E \mid E \text{ is an E-set of } \Delta\}$$

$$\mathcal{E}(\Delta) = \{E \mid \exists D' \subseteq D \text{ and } E \text{ is a R-extension for } (D', W)\}$$

The theorem below relates *E*-sets and R-extensions (for a proof, see the appendix).

Theorem 6:

Let Δ be a default theory. Then $E(\Delta) = \mathcal{E}(\Delta)$.

We are now ready to introduce the notion of C-extension:

Definition 3:

Let $\Delta = (D, W)$ be a default theory. A *C-extension for* Δ is a maximal element in $E(\Delta)$ with respect to the relation $\subseteq$.

The next theorems summarize the basic properties of C-extensions.

Theorem 7:

Let $\Delta=(D,W)$ be a default theory.

(a) If W is inconsistent, then $\mathrm{Th}(W)$ is the only C-extension of Δ.

(b) Δ always has at least one C-extension *(Existence of C-extensions)*.

(c) For any two C-extensions E and F for Δ, if $E\subseteq F$ then $E=F$ *(Minimality of C-extensions)*

Proof

Let Δ be a default theory.

(a) Immediate from Theorem 6.

(b) Note that $\mathcal{E}(\Delta)\neq\emptyset$, since $\mathrm{Th}(W)\in\mathcal{E}(\Delta)$. Thus, by Zorn's Lemma, there is a $\subseteq$-maximal element in $\mathcal{E}(\Delta)$. By Theorem 6, $\mathcal{E}(\Delta)=E(\Delta)$. Then, Δ has a C-extension.

(c) Let E and F be two C-extensions for Δ. If $E\subseteq F$ then $E=F$, since E and F are $\subseteq$-maximal elements in $E(\Delta)$.

□

Theorem 8: *(Strong minimality of C-extensions)*

Let $\Delta=(D,W)$ and $\Delta'=(D',W')$ be two default theories such that $D'\subseteq D$ and $W'\subseteq W$. For any C-extension E for Δ and any C-extension E' for Δ', if $E\subseteq E'$ then $E=E'$.

Proof

Let $\Delta=(D,W)$ and $\Delta'=(D',W')$ be two default theories such that $D'\subseteq D$ and $W'\subseteq W$. Let E be a C-extension for Δ and E' be a C-extension for Δ' and suppose that $E\subseteq E'$. By definition, $E'=\bigcup_{i=0}^{\infty} F_i$ where:

(1a) $\quad F_0=W'$

and, for $i\geq 0$:

(1b) $\quad F_{i+1} = \mathrm{Th}(F_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D'$ where $\alpha\in F_i$, $\neg\beta_1,\ldots,\neg\beta_m\notin E'$ and $\omega\in E'\}$

Construct E_i as:

(2a) $\quad E_0=W$

and, for $i\geq 0$:

(2b) $\quad E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D$ where $\alpha \in E_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin E'$ and $\omega \in E'\}$

We first prove that $E' = \bigcup_{i=0}^{\infty} F_i \subseteq \bigcup_{i=0}^{\infty} E_i$ by showing that $F_j \subseteq E_j$, for all $j \geq 0$. For $j=0$, we trivially have $F_0 \subseteq E_0$. Let $i \geq 0$ and assume that $F_i \subseteq E_i$. We will prove that $F_{i+1} \subseteq E_{i+1}$. Let $\omega \in F_{i+1}$. If $\omega \in Th(F_i)$, then $\omega \in E_{i+1}$ since $F_i \subseteq E_i$, by assumption, and $Th(E_i) \subseteq E_{i+1}$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$ such that $\alpha \in F_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin E'$ and $\omega \in E'$. As $F_i \subseteq E_i$, α is also in E_i. By (2b), if $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D$, since $D' \subseteq D$, $\alpha \in E_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin E'$ and $\omega \in E'$, then $\omega \in E_{i+1}$. Thus $F_{i+1} \subseteq E_{i+1}$. Hence $E' = \bigcup_{i=0}^{\infty} F_i \subseteq \bigcup_{i=0}^{\infty} E_i$.

We now prove that $\bigcup_{i=0}^{\infty} E_i \subseteq E'$. For $i=0$, we trivially have $E_0 \subseteq E'$, since $E_0 = W$ and $W \subseteq E \subseteq E'$, by assumption. The induction step follows because, by definition (2b), for all $i > 0$, $E_i \subseteq E'$. As $E' \subseteq \bigcup_{i=0}^{\infty} E_i \subseteq E'$, we conclude that $\bigcup_{i=0}^{\infty} E_i = E'$. By Definition 2, E' is then an E-set for Δ. Hence, $E' \in \mathcal{E}(\Delta)$. But, since $E \subseteq E'$ and the C-extension E is a $\subseteq$-maximal element in $\mathcal{E}(\Delta)$, then $E = E'$.

□

Note that item (c) of Theorem 7 can also be obtained as a corollary of the above theorem.

Theorem 9: *(Semi-monotonicity of C-extensions)*

Let $\Delta = (D,W)$ and $\Delta' = (D',W')$ be two default theories such that $D' \subseteq D$ and $W' = W$. For any C-extension E' for Δ', there is a C-extension E for Δ such that $E' \subseteq E$.

Proof

Let $\Delta = (D,W)$ and $\Delta' = (D',W)$ be two default theories such that $D' \subseteq D$. Since $D' \subseteq D$, we have $\mathcal{E}(\Delta') \subseteq \mathcal{E}(\Delta)$. Hence, by Theorem 6, $E(\Delta') \subseteq E(\Delta)$. Let E' be a C-extension of Δ'. Then, $E' \in E(\Delta')$ and, hence, $E' \in E(\Delta)$. Therefore, there is $E \in E(\Delta)$ such that $E' \subseteq E$ and E is $\subseteq$-maximal. Thus, there is a C-extension E of Δ such that $E' \subseteq E$.

□

The next two results compare C-extensions and R-extensions.

Theorem 10:

Let Δ be a default theory. If E is a R-extension for Δ then E is a C-extension for Δ.

Proof

Let Δ be a default theory. By the strong minimality of R-extensions, Theorem 2, if E is a R-extension for Δ then E is a $\subseteq$-maximal element in $\mathcal{E}(\Delta)$. Since $\mathcal{E}(\Delta) = E(\Delta)$, by Theorem 6, E is also a $\subseteq$-maximal element in $E(\Delta)$. Hence E is a C-extension for Δ.

□

The next theorem goes further and establishes that C-extensions and R-extensions are equivalent for normal default theories.

Theorem 11:

Let Δ be a normal default theory. A set of sentences E is a C-extension for Δ iff E is a R-extension for Δ.

Proof

Let $\Delta = (D, W)$ be a normal default theory. By Theorem 8, it suffices to prove that, if E is a C-extension for Δ then E is a R-extension for Δ. Let E be a C-extension for Δ. Then, E is a $\subseteq$-maximal element in $E(\Delta)$. Since $E(\Delta) = \mathcal{E}(\Delta)$, by Theorem 6, E is also in $\mathcal{E}(\Delta)$. By semi-monotonicity of R-extensions, there is a R-extension F for Δ such that $E \subseteq F$. By definition of $\mathcal{E}(\Delta)$, we have $F \in \mathcal{E}(\Delta)$. Since E is a $\subseteq$-maximal element in $\mathcal{E}(\Delta)$, we obtain $E = F$. Hence, E is a R-extension for Δ.

□

4. A PROOF THEORY FOR ARBITRARY DEFAULT THEORIES

In this section we introduce a proof theory for arbitrary default theories which is sound and complete with respect to C-extensions

Definition 4:

Let $\Delta=(D,W)$ be a default theory and γ be a sentence. A finite sequence $<D_0,...,D_k>$ of finite subsets of D is a *C-default proof* of γ from Δ iff:

P$_1$. $W \cup \text{CONSEQ}(D_0) \vdash \gamma$

P$_2$. For $1 \leq i \leq k$, $W \cup \text{CONSEQ}(D_i) \vdash \text{PREREQ}(D_{i-1})$

P$_3$. $D_k=\emptyset$

P$_4$. If W is consistent, then $W \cup \bigcup_{i=0}^{k} \text{CONSEQ}(D_i) \cup \{\beta\}$ must be consistent, for each $\beta \in \bigcup_{i=0}^{k} \text{JUSTIF}(D_i)$.

This definition deserves several brief comments. As for Definition 1, the proviso in P$_4$ that W be consistent is just a technical convenience and it implies that any finite sequence $<D_0,...,D_k>$ of finite subsets of D such that $D_k=0$ will be a proof of any sentence α from $\Delta=(D,W)$, if W is inconsistent. In fact, Definition 1 and Definition 4 will coincide for normal default theories since, in this case, each default has its justification identical to its consequent. Hence, our notion of default proof for arbitrary default theories extends Reiter's notion for normal default theories. But, more importantly, the definition of C-default proof is coherent with the definition of C-extension as shown by the following two theorems (whose proofs are in the appendix).

Theorem 12: *(Soundness of C-Default Proofs)*

Let Δ be a default theory and let γ be a sentence. If γ has a C-default proof from Δ, then Δ has a C-extension E such that $\gamma \in E$.

Theorem 13: *(Completeness of C-Default Proofs)*

Let Δ be a default theory and γ be a sentence. If Δ has a C-extension E such that $\gamma \in E$, then γ has a C-default proof from Δ.

Once such results have been established, it is straighforward to extend Reiter's concept of top down default proof, based on linear resolution or on another sound and complete refutation proof procedure, to arbitrary default theories. Finally, we note that the proofs of Theorem 12 and Theorem 13 follow a path different from the proofs of the equivalent theorems for R-default proofs found in Reiter [1980]. In fact, if the reader scrutinizes the proof of Theorem 13, he will perceive that it is in some sense a way of using the first-order Compactness Theorem to obtain a completeness result for C-default proofs.

5. NECESSARY CONDITIONS FOR A PROOF THEORY

Given an arbitrary concept of extension, in this section we describe necessary conditions for the existence of a proof theory which is sound and complete with respect to the given notion of extension.

We will adopt the following notation:

$\mathscr{L}$ a first-order language
$\boldsymbol{S}$ the set of all sentences of $\mathscr{L}$
$\wp$ the powerset of the set of all theories over $\mathscr{L}$
$\boldsymbol{D}$ the set of all closed defaults over $\mathscr{L}$
$\boldsymbol{D}^*$ the set of all finite sequences of finite sets of closed defaults over $\mathscr{L}$
λ the empty sequence in $\boldsymbol{D}^*$
$\mathbb{C}$ a set of closed default theories over $\mathscr{L}$

Definition 5:

An *extension operator* for $\mathbb{C}$ is a function $\Gamma:\mathbb{C}\rightarrow\wp$, that is, a function that maps each default theory in $\mathbb{C}$ into a set of theories over $\mathscr{L}$. Given a default theory $\Delta\in\mathbb{C}$, each $E\in\Gamma(\Delta)$ is called a Γ-*extension* for Δ.

Let $\bigcup\Gamma(\Delta)$ denote the union of all theories in $\Gamma(\Delta)$.

Definition 6:

Let Γ be an extension operator for $\mathbb{C}$.

(a) Γ satisfies the *existence property* iff, for all $\Delta\in\mathbb{C}$, then $\Gamma(\Delta)\neq\emptyset$.

(b) Γ satisfies the *strong existence property* iff, for all $(D,W)\in\mathbb{C}$, $\mathrm{Th}(W)\subseteq\bigcup\Gamma((D,W))$.

(c) Γ is *semi-monotonic* iff, for all $(D,W)\in\mathbb{C}$ and $(D',W)\in\mathbb{C}$, if $D'\subseteq D$ then, for all $E'\in\Gamma((D',W))$, there is $E\in\Gamma((D,W))$ such that $E'\subseteq E$.

(d) Γ is *weakly semi-monotonic* iff, for all $(D,W)\in\mathbb{C}$ and $(D',W)\in\mathbb{C}$, if $D'\subseteq D$ then $\bigcup\Gamma((D',W))\subseteq\bigcup\Gamma((D,W))$.

Proposition 1:

Let Γ be an extension operator for $\mathbb{C}$.

(a) If Γ is semi-monotonic then Γ is weakly semi-monotonic.

(b) if Γ satisfies the strong existence property then Γ satisfies the existence property.

Proof

(a) Immediate from Definition 6.

(b) Suppose that Γ satisfies the strong existence property. Let $(D,W)\in\mathbb{C}$. Then, $\mathrm{Th}(W)\subseteq\bigcup\Gamma((D,W))$. Hence, $\Gamma((D,W))\neq\emptyset$, since $\mathrm{Th}(W)\neq\emptyset$, because $\mathrm{Th}(W)$ is a theory and hence contains all tautologies. Therefore, Γ satisfies the existence property.

□

Definition 7:

A *test function* for $\mathbb{C}$ is a function $\varphi:\mathbb{C}\times \boldsymbol{D}^*\rightarrow\{0,1\}$ such that $\varphi((\emptyset,W),\lambda)=1$, for all $(\emptyset,W)\in\mathbb{C}$.

Definition 8:

Let φ be a test function in $\mathbb{C}$, Δ a default theory in $\mathbb{C}$ and β a sentence in $\boldsymbol{S}$. A sequence $<D_0,\ldots,D_k>$ in $\boldsymbol{D}^*$ is a *φ-default proof* of β from Δ iff

P_0. $\bigcup_{j=0}^{k} D_j\subseteq D$;

P_1. $W \cup \mathrm{CONSEQ}(D_0) \vdash \beta$;

P_2. For $1\leq i\leq k$, $W \cup \mathrm{CONSEQ}(D_i) \vdash \mathrm{PREREQ}(D_{i-1})$;

P_3. $D_k=\emptyset$;

P_4. $\varphi((D,W),<D_0\ldots D_k>)=1$.

The next definition introduces the concept of semi-locality, which requires that the test funtion be independent of the set of defaults not used in a proof.

Definition 9:

A test function φ for $\mathbb{C}$ is *semi-local* iff, for all (D,W), $(D',W)\in\mathbb{C}$, for all $d\in\boldsymbol{D}^*$, $\varphi((D,W),d) = \varphi((D',W),d)$.

By Definition 7, the empty sequence is then a φ-default proof of any sentence $\alpha\in\mathrm{Th}(W)$ from $\Delta=(\emptyset,W)$ since $\varphi((\emptyset,W),\lambda)=1$. Furthermore, if φ is semi-local, this observation extends to any default theory $\Delta=(D,W)$.

Definition 10:

Let φ be a test function for $\mathbb{C}$ and let Γ be an extension operator for $\mathbb{C}$.

(a) φ is *correct* with respect to Γ iff, for all $\Delta \in \mathbb{C}$ and for all $\beta \in S$, if there is a φ-default proof of β from Δ, then there is a Γ-extension E of Δ such that $\beta \in E$;

(b) φ is *complete* with respect to Γ iff, for all $\Delta \in \mathbb{C}$ and for all $\beta \in S$, if there is a Γ-extension E of Δ such that $\beta \in E$, then there is a φ-default proof of β from Δ.

The major result of this section (whose proof is in the appendix) states that the existence and the weakly semi-monotonicity properties of an extension operator Γ are necessary conditions for the existence of a default proof theory which is semi-local, sound and complete with respect to the extension operator Γ.

Theorem 14:

Let φ be a test function for $\mathbb{C}$ and let Γ be an extension operator for $\mathbb{C}$.

(a) If φ is semi-local and if φ is correct and complete with respect to Γ, then Γ is weakly semi-monotonic;

(b) If φ is semi-local and if φ is correct with respect to Γ, then Γ satisfies the strong existence property.

We note that the above conditions are necessary, but not sufficient. For example, the next example exhibits a semi-monotonic extension operator for which there is no complete default proof theory.

Example 1:

Let $\mathbb{C}$ be the class of all arbitrary (closed) default theories. Let Γ be an extension operator in $\mathbb{C}$ such that Γ maps Δ into a set containing a single inconsistent theory, for every Δ in $\mathbb{C}$. Clearly, Γ is semi-monotonic and, hence, weakly semi-monotonic. However, there is no test function φ which is complete with respect to Γ. Indeed, let $\Delta = (\emptyset, W)$ be a default theory in $\mathbb{C}$ where W is consistent. By assumption, **false** is in the unique element of $\Gamma(\Delta)$. However, we cannot prove **false** from Δ, since P_1 in Definition 8 is not satisfied because W is consistent and $D_0 = \emptyset$, as Δ has an empty set of defaults.

We conclude by relating the concepts introduced in this section with those described in sections 2, 3 and 4.

Consider first:

$\mathbb{R}$ the set of all closed normal default theories
r the extension operator for $\mathbb{R}$ induced by the notion of R-extension
ρ the test function for $\mathbb{R}$ induced by the test in P_4 of Definition 1

Then, by the results in section 2, ***r*** satisfies the existence property and is semi-monotonic, ρ is semi-local, the concept of R-default proof becomes a special case of Definition 8 and ρ is correct and complete with respect to ***r***.

Consider now:

$\mathbb{Z}$ the set of all closed (arbitrary) default theories
c the extension operator for $\mathbb{Z}$ induced by the notion of C-extension
γ the test function for $\mathbb{Z}$ induced by the test in P_4 of Definition 4

Then, by the results in section 3 and 4, ***c*** satisfies the existence property and is semi-monotonic, γ is semi-local, the concept of C-default proof becomes a special case of Definition 8 and γ is correct and complete with respect to ***c***.

6. CONCLUSIONS

We first defined a new concept of extension, called C-extension, using the idea of ignoring defaults when strictly necessary. We proved that C-extensions have interesting properties, such as existence, semi-monotonicity and minimality. We have also shown that C-extensions are equivalent to R-extensions for normal default theories. We then introduced the notion of C-default proof and established that this notion is sound and complete with respect to C-extensions. Finally, given an arbitrary concept of extension, we proved that the existence and weak semi-monotonicity properties are necessary conditions for the definition of a sound and complete proof theory.

Finally, we observe that the notions of C-extension and C-default proof can be extended to arbitrary open default proofs along the lines of section 7 of Reiter [1980].

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APPENDIX

Theorem 6:

Let Δ be a default theory. Then $E(\Delta) = \mathcal{E}(\Delta)$.

Proof

Let $\Delta = (D, W)$ be a default theory. We will prove that $E(\Delta) = \mathcal{E}(\Delta)$.

We first show that $E(\Delta) \subseteq \mathcal{E}(\Delta)$. Let $S \in E(\Delta)$ and let $D' = \mathrm{EGD}(S, \Delta)$. If we prove that S is a R-extension for (D', W), then we may conclude that $S \in \mathcal{E}(\Delta)$.

By Definition 2 and since $D' = \mathrm{EGD}(S, \Delta)$, we have that $S = \bigcup_{i=0}^{\infty} F_i$, where:

$$F_0 = W$$

and, for $i \geq 0$:

$$F_{i+1} = Th(F_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D' \text{ where } \alpha \in F_i,\ \neg\beta_1,\ldots,\neg\beta_m \notin S \text{ and } \omega \in S\}$$

Construct E_i as follows:

$$E_0 = W$$

and, for $i \geq 0$:

$$E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D' \text{ where } \alpha \in E_i \text{ and } \neg\beta_1,\ldots,\neg\beta_m \notin S\}$$

We prove that $F_j = E_j$, for all $j \geq 0$. For $j = 0$, trivially $F_0 = E_0$. Let $i \geq 0$ and assume that $F_i = E_i$. We will prove that $F_{i+1} = E_{i+1}$.

Let $\omega \in F_{i+1}$. If $\omega \in Th(F_i)$, then $\omega \in E_{i+1}$ since $F_i = E_i$, by assumption, and $Th(E_i) \subseteq E_{i+1}$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$ such that $\alpha \in F_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin S$ and $\omega \in S$. As $F_i = E_i$, α is also in E_i. But, if $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$, $\alpha \in E_i$ and $\neg\beta_1,\ldots,\neg\beta_m \notin S$, then $\omega \in E_{i+1}$. Hence, we may conclude that $F_{i+1} \subseteq E_{i+1}$.

Let $\omega \in E_{i+1}$. If $\omega \in Th(E_i)$, then $\omega \in F_{i+1}$ since $F_i = E_i$, by assumption, and $Th(F_i) \subseteq F_{i+1}$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$ such that $\alpha \in E_i$ and $\neg\beta_1,\ldots,\neg\beta_m \notin S$. Since $F_i = E_i$, α is also in F_i and, since $D' = EGD(S,\Delta)$, $\omega \in S$. But, if $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$, $\alpha \in F_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin S$ and $\omega \in S$, then $\omega \in F_{i+1}$. Hence, we may conclude that $E_{i+1} \subseteq F_{i+1}$.

Thus, we obtain that $F_j = E_j$, for all $j \geq 0$. Then, $\bigcup_{i=0}^{\infty} F_i = \bigcup_{i=0}^{\infty} E_i = S$. By Theorem 1, S is a R-extension for (D',W). Hence, $S \in \mathcal{E}(\Delta)$. Therefore, we may conclude that $E(\Delta) \subseteq \mathcal{E}(\Delta)$.

We now show that $\mathcal{E}(\Delta) \subseteq E(\Delta)$. Let $S \in \mathcal{E}(\Delta)$. Then, S is a R-extension for $\Delta' = (D',W)$, for some $D' \subseteq D$.

By Theorem 1, $S = \bigcup_{i=0}^{\infty} E_i$, where:

$$E_0 = W$$

and, for $i \geq 0$:

$$E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D' \text{ where } \alpha \in E_i \text{ and } \neg\beta_1,\ldots,\neg\beta_m \notin S\}$$

Construct F_i as follows:

$$F_0 = W$$

and, for $i \geq 0$:

$$F_{i+1} = Th(F_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D \text{ where } \alpha \in F_i,\ \neg\beta_1,\ldots,\neg\beta_m \notin S \text{ and } \omega \in S\}$$

We will prove that $S = F$, where $F = \bigcup_{i=1}^{\infty} F_i$.

We first prove that $S \subseteq F$ by showing that $E_j \subseteq F_j$, for all $j \geq 0$. For $j = 0$, we trivially have $E_0 \subseteq F_0$. Let $i \geq 0$ and assume that $E_i \subseteq F_i$. We will prove that $E_{i+1} \subseteq F_{i+1}$. Let $\omega \in E_{i+1}$. If $\omega \in Th(E_i)$, then $\omega \in F_{i+1}$ since $E_i \subseteq F_i$, by assumption, and $Th(F_i) \subseteq F_{i+1}$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$ such that $\alpha \in E_i$ and $\neg\beta_1,\ldots,\neg\beta_m \notin S$. As $E_i \subseteq F_i$, α is also in F_i. Since $\omega \in E_{i+1}$ and $E_{i+1} \subseteq S$, we have $\omega \in S$. But, if $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D'$, $D' \subseteq D$, $\alpha \in F_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin S$ and $\omega \in S$, then $\omega \in F_{i+1}$. Thus, $E_{i+1} \subseteq F_{i+1}$. Hence, we may conclude that $E_j \subseteq F_j$, for all $j \geq 0$, which implies that $S \subseteq F$.

We now prove that $F \subseteq S$, by showing that $F_j \subseteq S$, for all $j \geq 0$. For $j = 0$, we trivially have $F_0 = E_0 \subseteq S$. Let $i \geq 0$ and assume that $F_i \subseteq S$. We will prove that $F_{i+1} \subseteq S$. First observe that, since by assumption $F_i \subseteq S$ and since $Th(S) \subseteq S$, because S is a R-extension, it follows that $Th(F_i) \subseteq Th(S) = S$. Let $\omega \in F_{i+1}$. If $\omega \in Th(F_i)$, then $\omega \in S$. Otherwise, there is a default $(\alpha:\beta_1,\ldots,\beta_m/\omega) \in D$ such that $\alpha \in F_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin S$ and $\omega \in S$. Therefore, in both cases, $\omega \in S$. Then, $F_{i+1} \subseteq S$. Hence, we may conclude that $F_j \subseteq S$, for all $j \geq 0$, which implies that $F \subseteq S$.

Thus, we obtain $S = F = \bigcup_{i=0}^{\infty} F_i$. Therefore, by Definition 2, $S \in E(\Delta)$. Hence, we may conclude that $\mathcal{E}(\Delta) \subseteq E(\Delta)$.

□

Theorem 12: *(Soundness of C-Default Proofs)*

Let Δ be a default theory and let γ be a sentence. If γ has a C-default proof from Δ, then Δ has a C-extension E' such that $\gamma \in E'$.

Proof

Let $\Delta = (D, W)$ be a default theory and let γ be a sentence.

Suppose that W is inconsistent. Then, the result follows because $Th(W)$ is the only C-extension of Δ, by Theorem 7(a). Hence, $\gamma \in Th(W)$ trivially, because $Th(W)$ is an inconsistent theory.

Suppose now that W is consistent and that γ has a C-default proof $\langle D_0,\ldots,D_k \rangle$ from Δ. We will prove that γ belongs to a C-extension for Δ.

Define a set E as:

(1) $E = Th(W \cup \bigcup_{i=0}^{k} CONSEQ(D_i))$

Note that $\gamma \in E$, by P_1 of Definition 4.

Construct E_i as:

(2a) $E_0 = W$

and, for $i \geq 0$:

(2b) $E_{i+1} = Th(E_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta,}{\omega} \in D$ where $\alpha \in E_i$, $\neg\beta_1,\ldots,\neg\beta_m \notin E$ and $\omega \in E\}$

If we prove that $E = \bigcup_{i=0}^{\infty} E_i$, we may conclude that E is an E-set for Δ. Then, there is a C-extension E' for Δ such that $E \subseteq E'$. Since $\gamma \in E$, we obtain $\gamma \in E'$ and the result holds.

We can easily show, by induction, that $E_i \subseteq E$, for all $i \geq 0$, Then, $\bigcup_{i=0}^{\infty} E_i \subseteq E$.

To prove that $E \subseteq \bigcup_{i=0}^{\infty} E_i$, we first show that, for all $r \in [0,k]$:

(3) $W \cup \bigcup_{i=k-r}^{k} \mathrm{CONSEQ}(D_i) \subseteq E_{2r}$

Indeed, for $r=0$, (3) holds because $W = E_0$ and $D_k = \emptyset$. Let $s \in [0,k)$ and assume that (3) holds for $r=s$. We show that (3) holds for $r=s+1$.

Since by assumption (3) holds for $r=s$, we just have to prove that $\mathrm{CONSEQ}(D_{k-s-1}) \subseteq E_{2s+2}$. By P_2 of Definition 4:

(4) $W \cup \mathrm{CONSEQ}(D_{k-s}) \vdash \mathrm{PREREQ}(D_{k-s-1})$

Then, again since (3) holds for $r=s$, by (4) we have:

(5) $E_{2s} \vdash \mathrm{PREREQ}(D_{k-s-1})$

Hence, by (2b). that is by definition of E_{2s+1}, (5) implies that:

(6) $\mathrm{PREREQ}(D_{k-s-1}) \subseteq E_{2s+1}$

Now, since by assumption W is consistent, by P_4 of Definition 4, we have:

(7) $W \cup \bigcup_{i=0}^{k} \mathrm{CONSEQ}(D_i) \cup \{\beta\}$ is consistent, for all $\beta \in \mathrm{JUSTIF}(D_{k-s-1})$

Hence, by (1) and (7), we have:

(8) $\neg\beta \notin E$, for all $\beta \in \mathrm{JUSTIF}(D_{k-s-1})$

Furthermore, by (1), we also have:

(9) $\mathrm{CONSEQ}(D_{k-s-1}) \subseteq E$

Therefore, by (6), (8), (9) and (2b), that is, by definition of E_{2s+2}, we have:

(10) $\mathrm{CONSEQ}(D_{k-s-1}) \subseteq E_{2s+2}$.

Thus, we have just established that (3) holds for $r=s+1$. Therefore, (3) holds for all $r \in [0,k]$. In particular, (3) holds for $r=k$:

(11) $W \cup \bigcup_{i=0}^{k} \mathrm{CONSEQ}(D_i) \subseteq E_{2k}$

Hence, by (11) and definition of E_{2k+1}, we have:

(12) $E = \mathrm{Th}(W \cup \bigcup_{i=0}^{k} \mathrm{CONSEQ}(D_i)) \subseteq E_{2k+1} \subseteq \bigcup_{i=0}^{\infty} E_i$

as was to be shown.

□

Theorem 13: *(Completeness of C-Default Proofs)*

Let Δ be a default theory and γ be a sentence. If Δ has a C-extension E such that $\gamma \in E$, then γ has a C-default proof from Δ.

Proof

Let $\Delta = (D, W)$ be a default theory and let γ be a sentence.

Suppose that W is inconsistent. Then, the empty sequence λ is a C-default proof of γ from Δ, by Definition 4.

Suppose now that W is consistent and that Δ has a C-extension E such that $\gamma \in E$. We will prove that there is a C-default proof of γ from Δ.

By definition, $E = \bigcup_{i=0}^{\infty} \mathrm{E}_i$, where:

(1a) $\mathrm{E}_0 = W$

and, for $i \geq 0$:

(1b) $\mathrm{E}_{i+1} = \mathrm{Th}(\mathrm{E}_i) \cup \{\omega \mid \frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D \text{ where } \alpha \in \mathrm{E}_i,\ \neg\beta_1,\ldots,\neg\beta_m \notin E \text{ and } \omega \in E\}$

Construct $\overline{D}_i$ as:

(2a) $\overline{D}_0 = \emptyset$

and, for $i \geq 0$:

(2b) $\overline{D}_{i+1} = \{\frac{\alpha:\beta_1,\ldots,\beta_m}{\omega} \in D \mid \alpha \in \mathrm{E}_i,\ \neg\beta_1,\ldots,\neg\beta_m \notin E \text{ and } \omega \in E\}$

Clearly, $\overline{D}_i \subseteq \overline{D}_{i+1}$, for all $i \geq 0$. We prove by induction that, for all $j \geq 0$:

(3) $\mathrm{Th}(\mathrm{E}_j) = \mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_j))$

For $j = 0$ we trivially have $\mathrm{Th}(\mathrm{E}_0) = \mathrm{Th}(W)$. Let $i \geq 0$ and assume that $\mathrm{Th}(\mathrm{E}_i) = \mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_i))$. By (1b) and (2b):

(4) $\mathrm{E}_{i+1} = \mathrm{Th}(\mathrm{E}_i) \cup \mathrm{CONSEQ}(\overline{D}_{i+1})$

Then, by (4) and the induction hypothesis:

(5) $\mathrm{Th}(E_{i+1}) = \mathrm{Th}(\mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_i)) \cup \mathrm{CONSEQ}(\overline{D}_{i+1}))$

Since $\overline{D}_i \subseteq \overline{D}_{i+1}$, we then have:

(6) $\mathrm{Th}(E_{i+1}) = \mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_{i+1}))$

Hence, we may conclude that (3) holds for all $j \geq 0$.

Since $\gamma \in E$, there is $r \geq 0$ such that $\gamma \in E_r$. Then, $\gamma \in \mathrm{Th}(E_r)$. Define, for all $i \in [0,r]$, $\widetilde{D}_i = \overline{D}_{r-i}$. We now show that $<\widetilde{D}_0,...,\widetilde{D}_r>$ satisfies P_1, P_2, P_3 and P_4 of Definition 4.

To obtain P_1, observe that, since $\gamma \in \mathrm{Th}(E_r)$ and using (3), we have:

(7) $W \cup \mathrm{CONSEQ}(\overline{D}_r) \vdash \gamma$

Then, since $\overline{D}_r = \widetilde{D}_0$, we have:

(8) $W \cup \mathrm{CONSEQ}(\widetilde{D}_0) \vdash \gamma$

As for P_2, for all $i \in [1,r]$, using the definition of $\widetilde{D}_i$, the definition of E_{r-i} and (3), we have:

(9) $\mathrm{PREREQ}(\widetilde{D}_{i-1}) = \mathrm{PREREQ}(\overline{D}_{r-i+1}) \subseteq E_{r-i} \subseteq \mathrm{Th}(E_{r-i}) =$

$= \mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_{r-i})) = \mathrm{Th}(W \cup \mathrm{CONSEQ}(\widetilde{D}_i))$.

Then, for all $i \in [1,r]$:

(10) $W \cup \mathrm{CONSEQ}(\widetilde{D}_i) \vdash \mathrm{PREREQ}(\widetilde{D}_{i-1})$

Now, P_3 follows trivially because $\widetilde{D}_r = \overline{D}_0 = \emptyset$.

Finally, we proceed as follows to obtain P_4. By (3) and the fact that $\overline{D}_i \subseteq \overline{D}_{i+1}$, for all $i \geq 0$, we have:

(11) $\mathrm{Th}(E_r) = \mathrm{Th}(W \cup \mathrm{CONSEQ}(\overline{D}_r)) =$

$\mathrm{Th}(W \cup \bigcup_{i=0}^{r} \mathrm{CONSEQ}(\overline{D}_i)) = \mathrm{Th}(W \cup \bigcup_{i=0}^{r} \mathrm{CONSEQ}(\widetilde{D}_i))$

Now, by (2b) and since $\widetilde{D}_i = \overline{D}_{r-i}$, for all $i \in [1,r]$, we have:

(12) $\neg\beta \notin E$, for all $\beta \in \bigcup_{i=0}^{r} \mathrm{JUSTIF}(\widetilde{D}_i)$

Since $\mathrm{Th}(E_r) \subseteq E$, by (11) and (12), we obtain:

(13) $W \cup \bigcup_{i=0}^{r} \mathrm{CONSEQ}(\widetilde{D}_i) \cup \{\beta\}$ is consistent, for all $\beta \in \bigcup_{i=0}^{r} \mathrm{JUSTIF}(\widetilde{D}_i)$

Therefore, since we assumed that W is consistent, P_4 follows from (13).

However, $<\tilde{D}_0,...,\tilde{D}_r>$ is not necessarily a sequence of finite subsets of D. Thus, we show that we can inductively construct a sequence $<D_0,...,D_r>$ of finite subsets of D such that, for all $i\in[0,r]$, $D_i\subseteq\tilde{D}_i$ and the sequence $<D_0,...,D_r>$ satisfies P_1, P_2, P_3 and P_4 of Definition 4.

Since (8) holds, the Compacity Theorem for first-order logic (see Enderton [1972]) assures the existence of a finite subset D_0 of $\tilde{D}_0$ such that:

(14) $W \cup \text{CONSEQ}(D_0) \vdash \gamma$

Let $s\in[0,r)$ and assume that we have constructed $<D_0,...,D_s>$, satisfying P_1 and P_2, such that $D_i\subseteq\tilde{D}_i$, for all $i\in[0,s]$. By (10), since $s<r$, we have:

(15) $W \cup \text{CONSEQ}(\tilde{D}_{s+1}) \vdash \text{PREREQ}(\tilde{D}_s)$,

Hence, since $D_s\subseteq\tilde{D}_s$, we have:

(16) $W \cup \text{CONSEQ}(\tilde{D}_{s+1}) \vdash \text{PREREQ}(D_s)$.

As D_s is finite, $\text{PREREQ}(D_s)$ is actually equivalent to a first-order sentence. Then, we can apply the Compacity Theorem again and obtain $D_{s+1}\subseteq\tilde{D}_{s+1}$ such that:

(17) $W \cup \text{CONSEQ}(D_{s+1}) \vdash \text{PREREQ}(D_s)$.

Thus, we constructed a sequence $<D_0,...,D_r>$ of finite subsets of D which satisfies P_1 and P_2 and is such that $D_i\subseteq\tilde{D}_i$, for all $i\in[0,r]$. As the sequence $<\tilde{D}_0,...,\tilde{D}_r>$ satisfies P_3 and P_4, and since $D_i\subseteq\tilde{D}_i$, for all $i\in[0,r]$, the sequence $<D_0,...,D_r>$ also satisfies P_3 and P_4.

Hence, $<D_0,...,D_r>$ is a C-default proof of γ with respect to Δ.

□

Theorem 14

Let φ be a test function for $\mathbb{C}$ and let Γ be an extension operator for $\mathbb{C}$.

(a) If φ is semi-local and if φ is correct and complete with respect to Γ, then Γ is weakly semi-monotonic;

(b) If φ is semi-local and if φ is correct with respect to Γ, then Γ satisfies the strong existence property.

Proof

Let $\mathbb{C}$ be a class of default theories, φ be a function test for $\mathbb{C}$ and Γ be an extension operator for $\mathbb{C}$.

Part (a): Assume that φ is semi-local and that φ is correct and complete with respect to Γ. Let $\Delta=(D,W)$ and $\Delta'=(D',W)$ be default theories in $\mathbb{C}$ and assume that $D'\subseteq D$. Let $E'\in\Gamma(\Delta')$ and let $\beta\in E'$. We will prove that there is $E\in\Gamma(\Delta)$ such that $\beta\in E$.

Since φ is complete with respect to Γ, there is a φ-default proof $\boldsymbol{d}$ of β from Δ'. We prove that $\boldsymbol{d}$ is also a φ-default proof of β from Δ. Indeed, consider the conditions of Definition 8. Then, P_0 holds since $D'\subseteq D$ implies that $\bigcup_{j=0}^{k} D_j\subseteq D'\subseteq D$. Now, P_1, P_2 and P_3 trivially hold because $\boldsymbol{d}$ is a φ-default proof of β from Δ'. Finally, to prove P_4, first observe that $\varphi((D',W),\boldsymbol{d})=1$ because $\boldsymbol{d}$ is a φ-default proof of β from Δ'. Now, since φ is semi-local by assumption, $\varphi((D,W),\boldsymbol{d})=\varphi((D',W),\boldsymbol{d})=1$.

Recall now that φ is correct with respect to Γ, by assumption. Thus, there is $E\in\Gamma(\Delta)$ such that $\beta\in E$, since we just proved that there is a φ-default proof of β from Δ.

Hence, we may conclude that, for all $(D',W)\in\mathbb{C}$ and all $(D,W)\in\mathbb{C}$, if $D'\subseteq D$ then $\bigcup\Gamma((D',W))\subseteq\bigcup\Gamma((D,W))$. That is, Γ is weakly semi-monotonic.

Part (b): Assume that φ is semi-local and that φ is correct with respect to Γ. Let $\Delta=(D,W)$ be a default theory in $\mathbb{C}$. Let $\beta\in\mathrm{Th}(W)$. We will prove that there is $E\in\Gamma(\Delta)$ such that $\beta\in E$.

We first prove that the empty sequence λ is a φ-default proof of β from Δ. Indeed, λ trivially satisfy conditions P_0, P_2 and P_3 of Definition 8. Since $W\vdash\beta$, condition P_1 also holds. To prove that λ satisfies P_4, first observe that, by Definition 7, we have $\varphi((\emptyset,W),\lambda)=1$. Now, since φ is semi-local by assumption, $\varphi((D,W),\lambda)=\varphi((\emptyset,W),\lambda)=1$. Thus, λ is a φ-default proof of β from Δ.

But φ is correct with respect to Γ. Hence, there is $E\in\Gamma(\Delta)$ such that $\beta\in E$.

Therefore, we conclude that, for all $(D,W)\in\mathbb{C}$, we have $\mathrm{Th}(W)\subseteq\bigcup\Gamma((D,W))$. Hence, Γ satisfies the strong existence property.

□